6.4.6. *Warning.* It is not uncommon in the later literature to incorrectly define coherent as finitely generated. Please only use the correct definition, as the wrong definition causes confusion. Besides doing this for the reason of honesty, it will also help you see what hypotheses are actually necessary to prove things. And that always helps you remember what the proofs are — and hence why things are true.

6.4.7. *Why coherence?* Proposition 6.4.3 is a good motivation for the definition of coherence: it gives a small (in a non-technical sense) abelian category in which we can think about vector bundles.

There are two sorts of people who should care about the details of this definition, rather than living in a Noetherian world where coherent means finite type. Complex geometers should care. They consider complex-analytic spaces with the classical topology. One can define the notion of coherent $\mathscr{O}_X$-module in a way analogous to this (see [**Se1**, Def. 2]). Then Oka's Theorem states that the structure sheaf of $\mathbb{C}^n$ (hence of any complex manifold) is coherent, and this is very hard (see [**GR**, §2.5] or [**Rem**, §7.2]).

The second sort of people who should care are the sort of arithmetic people who may need to work with non-Noetherian rings, see §3.6.21, or work in non-archimedean analytic geometry.

6.4.8. *Remark: Quasicoherent and coherent sheaves on ringed spaces.* We will discuss quasicoherent and coherent sheaves on schemes, but they can be defined more generally (see Exercise 6.3.B for quasicoherent sheaves, and [**Se1**, Def. 2] for coherent sheaves). Many of the results we state will hold in greater generality, but because the proofs look slightly different, we restrict ourselves to schemes to avoid distraction.

6.4.9. $\star\star$ *Coherence is not a good notion in smooth geometry.* The following example from B. Conrad shows that in quite reasonable (but less "rigid") situations, the structure sheaf is not coherent over itself. Consider the ring $\mathscr{O}_0$ of germs of smooth (C^∞) functions at $0 \in \mathbb{R}$, with coordinate x. Now $\mathscr{O}_0$ is a local ring. Its maximal ideal $\mathfrak{m}$ is generated by x. (Key idea: suppose $f \in \mathfrak{m}$, and suppose f has a representative defined on (ϵ, ϵ). Then for $t \in (-\epsilon, \epsilon)$, $f(t) = \int_0^t f'(u)\, du = t \int_0^1 f'(tv)\, dv$. By "differentiating under the integral sign" repeatedly, we may check that $\int_0^1 f'(tv)\, dv$ is smooth. We deal with the case $t = 0$ separately as usual.)

Let $\phi \in \mathscr{O}_0$ be the germ of a smooth function that is 0 for $x \leq 0$, and positive for $x > 0$ (such as $\phi(x) = e^{-1/x^2}$ for $x > 0$). Consider the map $\times\phi \colon \mathscr{O}_0 \to \mathscr{O}_0$. The kernel is the ideal I_ϕ of functions vanishing for $x \geq 0$. Clearly I_ϕ is nonzero (for example, $\phi(-x) \in I_\phi$), but as $\mathfrak{m} = (x)$, $I_\phi = xI_\phi$, so I_ϕ cannot be finitely generated or else Nakayama's Lemma 8.2.9 would be contradicted. (Essentially the same argument shows that the sheaf of smooth functions on $\mathbb{R}$ is not coherent.) This is why coherence has no useful meaning for smooth manifolds.

6.5 Algebraic aside: The Jordan-Hölder Theorem

The Jordan-Hölder theorem in group theory is part of a more fundamental and somehow simpler story. The Jordan-Hölder "yoga" is why you can often factor some sort of algebraic object into primes or irreducibles, uniquely (in an appropriate sense), where each prime/irreducible appears the same number of times no matter how you factor. From this point of view, it generalizes unique factorization of integers; well-definedness of dimension of vector spaces; classification of finitely-generated abelian groups; classificaiton of finitlely generated mdules over principal ideal domains unique factorization of ideals in a Dedekind domain; and the traditional Jordan-Hölder theorem in group theory.

We will be mostly interested in modules over a ring, but there is no harm in working in a general abelian category $\mathscr{C}$. (This can be readily generalized further, as in Exercise 6.5.G.)

We say an object $M \in \mathscr{C}$ is **simple** (or **irreducible**) if its only subobjects are 0 and itself. A **composition series**has a **(finite) composition series**. for M is a (finite) filtration

(6.5.0.1) $$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M$$

such that the quotients M_{i+1}/M_i are all simple. If M has a composition series, we say that M has **finite length**.

6.5.1. The Jordan-Hölder Theorem. — *If M has a finite composition series, then all composition series for M have the same length, and the quotients are all the same, possibly rearranged.*

6.5.2. *Definition.* We call the length of any of the composition series for M the **length** of M, denoted $\ell(M)$. This notion is well-defined by the Jordan-Hölder Theorem. But we even have a refined notion: we have the multiplicity with which each simple object appears in any composition series for M.

If M is not of finite length, we say $\ell(M) = \infty$.

6.5.3. *Example.* In the category of abelian groups, the finite-length objects are the *finite* abelian groups. The Jordan-Hölder Theorem in this case, applied to $\mathbb{Z}/n\mathbb{Z}$, can be used to give the unique factorization of n.

6.5.4. *Proof of the Jordan-Hölder Theorem 6.5.1.*

Suppose we have two finite composition series, (6.5.0.1) and

$$0 = M'_0 \subsetneq M'_1 \subsetneq \cdots \subsetneq M'_{n'-1} \subsetneq M'_{n'} = M,$$

of one object $M \in \mathscr{C}$. Make a rectangular array with entries $M_{i,j} := M'_i \cap M''_j$, as in Figure 6.1. Figure 6.2 shows this construction applied to two composition series for the $\mathbb{Z}$-module $\mathbb{Z}/(12)$,

$$(12) \subsetneq (6) \subsetneq (2) \subsetneq (1) \quad \text{and} \quad (12) \subsetneq (4) \subsetneq (2) \subsetneq (1).$$

6.5.5. Observe that

- $M_{i,j} \subset M_{i',j'}$ if $i \leq i'$ and $j \leq j'$,
- $M_{n,j} = M'_j$ and $M_{i,n'} = M_i$,
- $M_{0,j} = M_{i,0} = 0$, and
- $M_{i,j} \cap M_{i',j'} = M_{\min(i,i'),\min(j,j')}$.

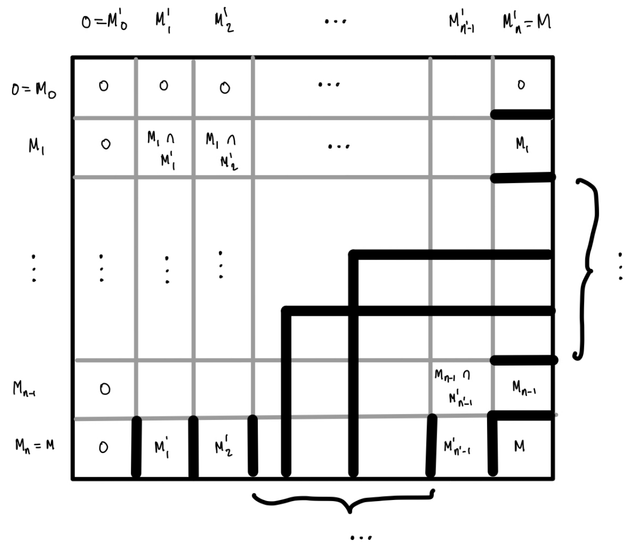


FIGURE 6.1. The rectangular array in the proof of the Jordan-Hölder Theorem

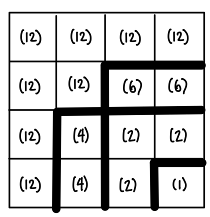


FIGURE 6.2. A sample "Jordan-Hölder table" for two composition series for $\mathbb{Z}/(12)$

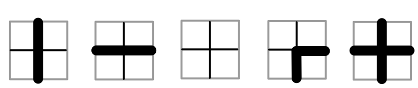
6.5.A. EXERCISE. Show (by descending induction on i) that $M_{i,j}/M_{i,j-1}$ is 0 or isomorphic to the simple element M'_j/M'_{j-1}.

Similarly, we have the analogous statement for $M_{i,j}/M_{i-1,j}$. In the rectangular array, draw a *thick line* between these two (horizontally or vertically adjacent) entries if the quotient is nonzero, and label that line with the (isomorphism class of) the simple group (again, see Figure 6.1).

Consider any 2×2 subsquare of the array:

$$\begin{array}{|c|c|} \hline M_{i,j} & M_{i,j+1} \\ \hline M_{i+1,j} & M_{i+1,j+1} \\ \hline \end{array} = \begin{array}{|c|c|} \hline A & B \\ \hline C & D \\ \hline \end{array}$$

We will see that the thick lines inside the square form one of the following five patterns (each of which appears in Figure 6.2).



From $A = B \cap C$, we see that (i) if $D = B$, then $A = C$, and (ii) if $D = C$, then $A = B$. That takes care of the first three cases.

Suppose next that both D/B and D/C are *both* nonzero (hence simple). If $B = C$, we are in the fourth ("elbow") case.

Finally, otherwise, we will see that $D/B \cong C/A$ (and similarly, $D/C \cong B/A$), and we are in the fifth case. Consider the map $C \to D/B$. Then A is precisely the kernel, from $A = B \cap C$ (§6.5.5). Thus we have an injection $C/A \hookrightarrow D/B$. By simplicity of D/B, either C/A is zero, or this injection is actually an isomorphism $C/A \xrightarrow{\sim} D/B$. □

6.5.B. EXERCISE. Prove the Jordan-Hölder Theorem. Hint: notice that the thickened lines can be interpreted as paths from the right side of the table to the bottom of the table, with one left turn. This will give a bijection between the simple quotients of one filtration, and the simple quotients of the other filtration.

6.5.C. EXERCISE. Show that every subquotient of a finite-length object M is finite length. Possible approach: suppose the subquotient if M''/M', where $M' \subseteq M'' \subseteq M$. Choose a composition series $M_\bullet$ for M. Make a new rectangular table in a similar way, using the composition series $M_\bullet$, and the filtration $0 \subseteq M' \subseteq M'' \subseteq M$. Think suitably about paths, similar to the proof of the Jordan-Hölder Theorem.

6.5.D. EXERCISE. Show that length is additive in exact sequences: if $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then $\ell(M) = \ell(M') + \ell(M'')$. (Do *not* assume these objects have finite length.)

6.5.E. EXERCISE. Show that any filtration of a finite length module can be refined into a composition series.

6.5.F. UNIMPORTANT EXERCISE. Show thow that the finite length objects in $\mathscr{C}$ form a full subcategory of $\mathscr{C}$.

The category of groups does not form an abelian category, so Theorem 6.5.1 can't immediately imply the traditional Jordan-Hölder Theorem for groups. However, the same proof applies without change, with only one additional input.

6.5.G. EXERCISE (THAT WE WON'T USE). Prove the Jordan-Hölder Theorem for groups. You will need the second isomorphism theorem: if N_1 and N_2 are normal subgroups, then N_1N_2 forms a *normal* subgroup, and $N_2/(N_1 \cap N_2) \cong (N_1N_2)/N_1$.

6.5.6. Additional facts particular to modules over a ring.

We now apply these concepts specifically to the category Mod_A.

6.5.H. EXERCISE. Show that the simple objects of Mod_A are precisely the objects of the form $A/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of A.

6.5.I. EXERCISE. Suppose M is a finite length A-module, and (6.5.0.1) is a composition series for M, with $M_i/M_{i-1} \cong A/\mathfrak{m}_i$ (where the $\mathfrak{m}_i$ are maximal ideals, not necessarily distinct). Show that M is annihilated by $\mathfrak{m}_1 \cdots \mathfrak{m}_n$. Equivalently, M is an $A/(\mathfrak{m}_1 \cdots \mathfrak{m}_n)$-module.

Suppose now that the list $(\mathfrak{m}_1, \dots, \mathfrak{m}_n)$ consists of the distinct maximal ideals $\mathfrak{n}_1, \dots, \mathfrak{n}_s$, appearing with multiplicity $\ell_1, \dots, \ell_s$. (These are the "refined" lengths mentioned in Definition 6.5.2.)

By the Chinese Remainder Theorem,

$$A/\mathfrak{m}_1 \dots \mathfrak{m}_n \cong A/\mathfrak{n}_1^{\ell_1} \cdots \mathfrak{n}_s^{\ell_s} \cong A/\mathfrak{n}_1^{\ell_1} \times \cdots A/\mathfrak{n}_s^{\ell_s}.$$

For $1 \leq i \leq s$, let $e_i \in A$ be an element of A so that $e_i \equiv 1 \pmod{\mathfrak{n}_i^{\ell_i}}$ and $e_i \equiv 0 \pmod{\mathfrak{n}_j^{\ell_j}}$ for $i \neq j$. The e_i exist by the Chinese Remainder Theorem.

6.5.J. EXERCISE. Show that $M \cong e_1 M \times \cdots \times e_s M$, and $e_i M$ is a finite-length module where all the simple quotients are $A/\mathfrak{m}_i$. Thus M is a product of pieces, each with composition series with only one type of "simple factor".

6.5.K. EXERCISE. Suppose M is a finite length A-module. Show that M is finitely generated, and $\operatorname{Supp} M$ consists of finitely many points of $\operatorname{Spec} A$, all closed. (The converse will be proved in Exercise 6.6.X(a).) We thus have a notion of the "length of M at each of these cloesd points".

6.5.7. *Applying this language to schemes.* We next consider the category $QCoh_X$ of quasicoherent sheaves on a scheme X. We have the notion of the length $\ell(\mathscr{F})$ of a finite-length quasicoherent sheaf on X.

6.5.L. EXERCISE. Show that the simple objects of $QCoh_X$ are the structure sheaves of closed points.

6.5.M. EXERCISE. Suppose that $\mathscr{F}$ is a finite length element of $QCoh_X$. Show that $\mathscr{F}$ is finite type, and $\operatorname{Supp} \mathscr{F}$ consists of finitely many points of X, all closed. (The converse will be proved in Exercise 6.6.X(b).) Explain how to define the length of a $\mathscr{F}$ at one of the points of $\operatorname{Supp} \mathscr{F}$.

6.5.8. *Definition.* The **length of a scheme** X is the length of the structure sheaf $\mathscr{O}_X$ (in $QCoh_X$). A scheme X is **finite length** or **Artinian** if $\mathscr{O}_X$ is finite length.

6.6 Visualizing schemes: Associated points and zerodivisors

The theory of *associated points* of a module refines the notion of support (§4.1.7). Associated points will help us understand and visualize nilpotents, and generalize the notion of "rational functions" to non-integral schemes. They are useful in ways we won't use, for example through their connection to primary decomposition. They might be most useful for us in helping us understand and visualize (non-)zerodivisors, which will come up repeatedly, through effective Cartier divisors and line bundles, regular sequences, depth and Cohen-Macaulayness, and more.

There is no particular reason to discuss associated points now, and this section can be read independently, at leisure. But it is a good opportunity to practice visualizing geometry, and to learn some useful algebra.

6.6.1. Motivation. Figure 6.3 is a sketch of a scheme X. We see two connected components, and three irreducible components. The irreducible components of X have