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**6.5.1. Motivation.** Figure 6.1 is a sketch of a scheme X. We see two connected components, and three irreducible components. The irreducible components of X have dimensions 2, 1, and 1, although we won't be able to make sense of "dimension" until Chapter 12. Both connected components are nonreduced.

We see a little more in this picture, which we will make precise in this section, in terms of "associated points". The reducible connected component seems to have different amounts of nonreduced behavior on different loci. The scheme X has six associated points, which are the generic points of the irreducible subsets "visible" in the picture. A function on X is a zerodivisor if its zero locus contains any of these six irreducible subvarieties.

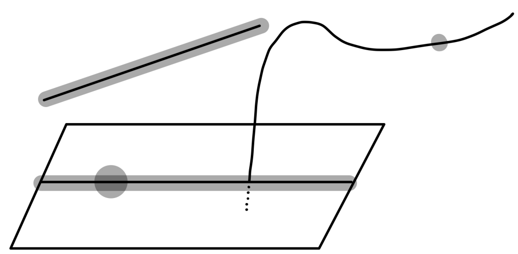

FIGURE 6.1. This scheme has six associated points, of which three are embedded points. A function is a zerodivisor if it vanishes at any of these six points.

Suppose M is a finitely generated module over a Noetherian ring A. For example, M could be A itself. Then there are some special points of Spec A that are particularly crucial to understanding M. These are the *associated points* of M (or equivalently, the *associated prime ideals* of M — we will use these terms interchangeably). As motivation, we give a zillion properties of associated points, and leave it to you to verify them from the theory developed in the rest of this section

As you read this section, you may wish to keep in mind

$$M = A = k[x,y]/(y^2, xy)$$

(Figure 4.4) as a running example.

**6.5.2. A zillion properties of associated points.** Here are some of the properties of associated points that we will prove.

There are finitely many associated points $\operatorname{Ass}_A M \subset \operatorname{Spec} A$.

The support of M is the closure of the associated points of M: $\operatorname{Supp} M = \overline{\operatorname{Ass}_A M}$. The support of any submodule of M is the closure of some subset of the associated points of $M$. The support of any element of $M$ is the closure of some subset of the associated points.

The associated points are precisely the generic points of irreducible components of $\operatorname{Supp} m$ for all $m \in M$. The associated points are precisely the generic points of those $\operatorname{Supp} m$ which are irreducible. The associated primes are precisely those prime ideals that are annihilators of some element of $M$.

Taking "associated points" commutes with localization. Hence this notion is "geometric in nature", which will (in §6.5.2) allow us to extend the notion to coherent sheaves on locally Noetherian schemes.

Associated points behave *fairly* well in exact sequences. For example, the associated points of a submodule are a subset of the associated points of the module.

If $I \subset A$ is an ideal, the associated primes $\mathfrak{p}$ of $A/I$ are precisely those $\mathfrak{p}$ such that a $\mathfrak{p}$-primary ideal appears in the primary decomposition of $I$.

We will repeatedly use the fact that *an element of* $A$ *is a zerodivisor if and only if it vanishes at an associated point.*

An element of $A$ is a unit if and only if it vanishes at no associated point . An element of $A$ is nilpotent if and only if it vanishes at every associated point.

The locus of points $[\mathfrak{p}]$ of $\operatorname{Spec} A$ where the stalk $A_{\mathfrak{p}}$ is nonreduced is the closure of some subset of the associated points.

An associated point that is in the closure of another associated point is said to be an *embedded point*. If $A$ is reduced, then $\operatorname{Spec} A$ has no embedded points. Hypersurfaces in $\mathbb{A}^n_k$ have no embedded points. We will later see that complete intersections have no embedded points (§29.2.7).

Elements of $M$ are determined by their localization at the associated points. Sections of the corresponding sheaf $\widetilde{M}$ (Exercise/Definition 4.1.D) are determined by their germs at the associated points.

This discussion immediately implies a notion of **associated point** for a coherent sheaf on a locally Noetherian scheme, with all the good properties described here. The phrase **associated point of a locally Noetherian scheme** $X$ (without explicit mention of a coherent sheaf) means "associated point of $\mathscr{O}_X$", and similarly for **embedded oints**.

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