By the Affine Communication Lemma 5.3.2, and Exercises 6.4.A, 6.4.B, and 6.4.C, it suffices to check "finite typeness" (resp. finite presentation, coherence) on the open sets in a single affine cover.

6.4.6. *Warning.* It is not uncommon in the later literature to incorrectly define coherent as finitely generated. Please only use the correct definition, as the wrong definition causes confusion. Besides doing this for the reason of honesty, it will also help you see what hypotheses are actually necessary to prove things. And that always helps you remember what the proofs are — and hence why things are true.

6.4.7. *Why coherence?* Proposition 6.4.3 is a good motivation for the definition of coherence: it gives a small (in a non-technical sense) abelian category in which we can think about vector bundles.

There are two sorts of people who should care about the details of this definition, rather than living in a Noetherian world where coherent means finite type. Complex geometers should care. They consider complex-analytic spaces with the classical topology. One can define the notion of coherent $\mathscr{O}_X$-module in a way analogous to this (see [**Se1**, Def. 2]). Then Oka's Theorem states that the structure sheaf of $\mathbb{C}^n$ (hence of any complex manifold) is coherent, and this is very hard (see [**GR**, §2.5] or [**Rem**, §7.2]).

The second sort of people who should care are the sort of arithmetic people who may need to work with non-Noetherian rings, see §3.6.21, or work in non-archimedean analytic geometry.

6.4.8. $\star\star$ *Coherence is not a good notion in smooth geometry.* The following example from B. Conrad shows that in quite reasonable (but less "rigid") situations, the structure sheaf is not coherent over itself. Consider the ring $\mathscr{O}_0$ of germs of smooth (C^∞) functions at $0 \in \mathbb{R}$, with coordinate x. Now $\mathscr{O}_0$ is a local ring. Its maximal ideal $\mathfrak{m}$ is generated by x. (Key idea: suppose $f \in \mathfrak{m}$, and suppose f has a representative defined on (ϵ, ϵ). Then for $t \in (-\epsilon, \epsilon)$, $f(t) = \int_0^t f'(u)\, du = t \int_0^1 f'(tv)\, dv$. By "differentiating under the integral sign" repeatedly, we may check that $\int_0^1 f'(tv)\, dv$ is smooth. We deal with the case $t = 0$ separately as usual.)

Let $\phi \in \mathscr{O}_0$ be the germ of a smooth function that is 0 for $x \leq 0$, and positive for $x > 0$ (such as $\phi(x) = e^{-1/x^2}$ for $x > 0$). Consider the map $\times\phi \colon \mathscr{O}_0 \to \mathscr{O}_0$. The kernel is the ideal I_ϕ of functions vanishing for $x \geq 0$. Clearly I_ϕ is nonzero (for example, $\phi(-x) \in I_\phi$), but as $\mathfrak{m} = (x)$, $I_\phi = xI_\phi$, so I_ϕ cannot be finitely generated or else Nakayama's Lemma 8.2.9 would be contradicted. (Essentially the same argument shows that the sheaf of smooth functions on $\mathbb{R}$ is not coherent.) This is why coherentce has no useful meaning for smooth manifolds.

6.5 Visualizing schemes: Associated points and zerodivisors

The *associated points* of a module refine the notion of support (§4.1.7). They will help us understand and visualize nilpotents, and generalize the notion of "rational functions" to non-integral schemes. They are useful in ways we won't use, for example through their connection to primary decomposition. They might be most useful for us in helping us understand and visualize (non-)zerodivisors, which will come up repeatedly, through effective Cartier divisors and line bundles, regular sequences, depth and Cohen-Macaulayness, and more.

There is no particular reason to discuss associated points now, and this section can be read independently, at leisure. We will not need this material in any essential way for some time, but it is a good opportunity to practice visualizing geometry, and to learn some useful algebra.

6.5.1. Motivation. Figure 6.1 is a sketch of a scheme X. We see two connected components, and three irreducible components. The irreducible components of X have dimensions 2, 1, and 1, although we won't be able to make sense of "dimension" until Chapter 12. Both connected components are nonreduced.

We see a little more in this picture, which we will make precise in this section, in terms of "associated points". The reducible connected component seems to have different amounts of nonreduced behavior on different loci. The scheme X has six associated points, which are the generic points of the irreducible subsets "visible" in the picture. A function on X is a zerodivisor if its zero locus contains any of these six irreducible subvarieties.

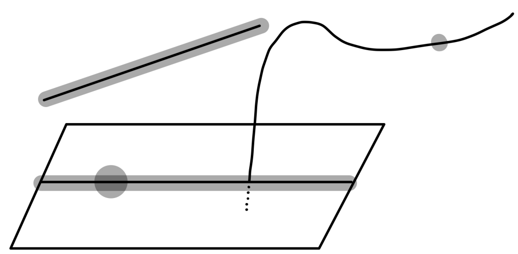

FIGURE 6.1. This scheme has six associated points, of which three are embedded points. A function is a zerodivisor if it vanishes at any of these six points.

Suppose M is a finitely generated module over a Noetherian ring A. For example, M could be A itself. Then there are some special points of $\operatorname{Spec} A$ that are particularly crucial to understanding M. These are the *associated points* of M (or equivalently, the *associated prime ideals* of M — we will use these terms interchangeably). As motivation, we give a zillion properties of associated points, and leave it to you to verify them from the theory developed in the rest of this section

As you read this section, you may wish to keep in mind

$$M = A = k[x,y]/(y^2, xy)$$

(Figure 4.4) as a running example.

6.5.2. A zillion properties of associated points. Here are some of the properties of associated points that we will prove.

There are finitely many associated points $\operatorname{Ass}_A M \subset \operatorname{Spec} A$.

The support of M is the closure of the associated points of M: $\operatorname{Supp} M = \overline{\operatorname{Ass}_A M}$. The support of any submodule of M is the closure of some subset of the associated points of M. The support of any element of M is the closure of some subset of the associated points.

The associated points are precisely the generic points of irreducible components of $\operatorname{Supp} m$ for all $m \in M$. The associated points are precisely the generic points of those $\operatorname{Supp} m$ which are irreducible. The associated primes are precisely those prime ideals that are annihilators of some element of M.

Taking "associated points" commutes with localization. Hence this notion is "geometric in nature", which will (in §6.5.2) allow us to extend the notion to coherent sheaves on locally Noetherian schemes.

Associated points behave *fairly* well in exact sequences. For example, the associated points of a submodule are a subset of the associated points of the module.

If $I \subset A$ is an ideal, the associated primes $\mathfrak{p}$ of A/I are precisely those $\mathfrak{p}$ such that a $\mathfrak{p}$-primary ideal appears in the primary decomposition of I.

We will repeatedly use the fact that *an element of A is a zerodivisor if and only if it vanishes at an associated point.*

An element of A is a unit if and only if it vanishes at no associated point . An element of A is nilpotent if and only if it vanishes at every associated point.

The locus of points $[\mathfrak{p}]$ of $\operatorname{Spec} A$ where the stalk $A_{\mathfrak{p}}$ is nonreduced is the closure of some subset of the associated points.

An associated point that is in the closure of another associated point is said to be an *embedded point*. If A is reduced, then $\operatorname{Spec} A$ has no embedded points. Hypersurfaces in $\mathbb{A}^n_k$ have no embedded points. We will later see that complete intersections have no embedded points (§29.2.7).

Elements of M are determined by their localization at the associated points. Sections of the corresponding sheaf $\widetilde{M}$ (Exercise/Definition 4.1.D) are determined by their germs at the associated points.

This discussion immediately implies a notion of **associated point** for a coherent sheaf on a locally Noetherian scheme, with all the good properties described here. The phrase **associated point of a locally Noetherian scheme** X (without explicit mention of a coherent sheaf) means "associated point of $\mathscr{O}_X$", and similarly for **embedded points**.

We now establish these zillion facts.

6.5.3. More on the notion of support.

The notion of *associated points* of an A-module M refines the notion of *support* (in the case where M is finitely generated over a Noetherian ring A). (In what follows, we make no assumptions that A is Noetherian or that M is finitely generated until we need to.) To set this up, recall (§4.1.7) that the *support of* $m \in M$,

$$\operatorname{Supp} m = \{[\mathfrak{p}] \in \operatorname{Spec} A : m_{\mathfrak{p}} \neq 0\},$$

is a closed subset, and thus of the form $V(I)$ for some I. Exercise 6.5.A gives the "best such" I. Define the **annihilator ideal** $\operatorname{Ann}_A m \subset A$ of an element m of an A-module M by:

$$\operatorname{Ann}_A m := \{a \in A : am = 0\} = \ker(A \xrightarrow{\times m} M).$$

The subscript A is omitted if it is clear from context.

6.5.A. Easy Important Exercise. Show that $\operatorname{Supp} m = V(\operatorname{Ann} m)$.

Recall (Definition 2.7.6) that

$$\operatorname{Supp} \widetilde{M} = \{\mathfrak{p} \in \operatorname{Spec} A : \widetilde{M}_{\mathfrak{p}} \neq 0\},$$

and the analogous Definition 4.1.7 of the *support of the module* M,

$$\operatorname{Supp} M := \{\mathfrak{p} \in \operatorname{Spec} A : M_{\mathfrak{p}} \neq 0\}, \tag{6.5.3.1}$$

so $\operatorname{Supp} M = \operatorname{Supp} \widetilde{M}$. If M is a principal module generated by $m \in M$, then

$$\operatorname{Supp} M = \operatorname{Supp} Am = \operatorname{Supp} m = V(\operatorname{Ann} m).$$

The notions of support and associated points behave well in exact sequences, and under localization. We begin to explain this now.

6.5.4. *The notion of support behaves well in exact sequences.*

6.5.B. EXERCISE. Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of A-modules.

(a) Show that $\operatorname{Supp} M = \operatorname{Supp} M' \cup \operatorname{Supp} M''$.
(b) Show that if M is a finitely generated module, then $\operatorname{Supp} M$ is a closed subset of $\operatorname{Spec} A$. (Hint: induction on the number of generators.)

Warning: $\operatorname{Supp} M$ need not be closed in general; consider $A = \mathbb{Z}$ and $M = \oplus_{p \text{ prime}} \mathbb{Z}/(p)$.

6.5.C. EXERCISE. Suppose M is a finitely generated A-module, and $x \in A$ has value 0 at all the points of $\operatorname{Supp} M$, i.e., x is contained in all of the primes where M is supported. Show that some power x^n of x annihilates every element of M. (Hint: annihilate a generating set.)

6.5.5. Definition: Associated points and associated primes.

Define the **associated prime ideals** of an A-module M to be those *prime* ideals of A of the form $\operatorname{Ann}_A(m)$ for some $m \in M$. Define the **associated points** of M to be the corresponding points of $\operatorname{Spec} A$; we use the terminology "associated points" and "associated primes" interchangeably. The set of associated points is denoted $\operatorname{Ass}_A M \subset \operatorname{Spec} A$. The subscript A is dropped if it is clear from the context. (To help remember the definition and the notation, some call these the *assassins*, as they are the primes that can ruthlessly annihilate elements of the module, without remorse. But we will not use this term.)

6.5.6. The **associated primes of a ring** A are the associated primes of A considered as an A-module (i.e., $M = A$).

6.5.D. EASY EXERCISE (ASSOCIATED POINTS OF INTEGRAL DOMAINS). If A is an integral domain, show that $\operatorname{Ass} A = \{[(0)]\}$ — the zero ideal is the only associated prime.

6.5.E. EXERCISE (ASSOCIATED POINTS OF HYPERSURFACES). Given $f \in k[x_1, \dots, x_n]$, show that the associated primes of $k[x_1, \dots, x_n]/(f)$ are those principal ideals generated by the prime factors of f. (Your argument will apply more generally to any $f \in A$ where A is a Unique Factorization Domain.)

6.5.7. *The observation that* $[\mathfrak{p}] \in \operatorname{Ass}_A(M)$ *if and only if there is an injection* $A/\mathfrak{p} \hookrightarrow M$ *of* A*-modules will be essential.* The next two exercises might drive this home.

6.5.F. EXERCISE. Suppose $M' \subset M$. Show that $\operatorname{Ass} M' \subset \operatorname{Ass} M$.

The corresponding statement for "support" is implicit in Exercise 6.5.B(a).

6.5.G. EXERCISE. Show that $\operatorname{Ass} M \subset \operatorname{Supp} M$.

If M is finitely generated, then $\operatorname{Supp} M$ is closed (Exercise 6.5.B), so $\overline{\operatorname{Ass} M} \subset \operatorname{Supp} M$. Equality will be shown in Proposition 6.5.24, when A is Noetherian.

6.5.8. Nonzero modules over Noetherian rings have associated points.
Suppose m is a nonzero element of an A-module M. Observe that for any nonzero multiple xm of m, $\operatorname{Ann} m \subseteq \operatorname{Ann} xm \subsetneq A$.

6.5.H. EXERCISE. Suppose A is Noetherian. Show that there is some multiple $n = xm$ such that *any* nonzero multiple $yn \neq 0$ of n satisfies $\operatorname{Ann} yn = \operatorname{Ann} n$.

6.5.9. Proposition. — *Continuing the notation of the previous exercise, we have that* $\operatorname{Ann} n$ *is a prime ideal.*

Proof. Suppose $ab \in \operatorname{Ann} n$, so $abn = 0$. Then either $bn = 0$ (in which case $b \in \operatorname{Ann} n$), or else $a \in \operatorname{Ann} bn = \operatorname{Ann} n$. □

We have thus proved the following.

6.5.10. Proposition (nonzero modules over Noetherian rings have associated primes). — *If* M *is a nonzero module over a Noetherian ring* A*, then* $\operatorname{Ass}_A M$ *is nonempty. More precisely, for any* $m \neq 0$ *in* M*, there is an associated prime* $\mathfrak{p}$ *containing* $\operatorname{Ann} m$*, and* $\mathfrak{p} = \operatorname{Ann} xm$ *for some* $x \in A$*.*

6.5.11. Localizations at the associated primes.
Recall the useful fact that $M \to \prod_{\mathfrak{p} \in \operatorname{Spec} A} M_{\mathfrak{p}}$ is an injection (Exercise 4.1.F). Our current situation is much better: we can take the product over only the localization at *associated* primes.

6.5.I. EXERCISE. Suppose M is a module over a Noetherian ring A. Show that the natural map

$$M \longrightarrow \prod_{\mathfrak{p} \in \operatorname{Ass} M} M_{\mathfrak{p}} \qquad (6.5.11.1)$$

is an injection. Hint: if the kernel K is nonzero, then K has an associated prime $\mathfrak{p}$, which is the annihilator of some $m \in K \subset M$, and m is nonzero in $M_{\mathfrak{p}}$.

Clearly we need only the maximal among the associated primes in (6.5.11.1).

6.5.12. Zerodivisors = elements of associated primes.

6.5.13. Proposition. — *Suppose* $f \in A$*, with* A *Noetherian. Then* f *is a zerodivisor on* M *if and only if* f *vanishes at an associated point of* M*. Translation: the set of zerodivisors is the union of the associated prime ideals.*

Again, we need only the maximal among the associated primes. For example, if $(A, \mathfrak{m})$ is a local ring, then $\mathfrak{m}$ is an associated prime if and only if every element of $\mathfrak{m}$ is a zerodivisor.

Proof. Suppose f vanishes at an associated point $[\mathfrak{p}]$ of M. Choose m with $\operatorname{Ann} m = \mathfrak{p}$. Then $fm = 0$ while $m \neq 0$, so f is a zerodivisor.

Conversely, if f is vanishes at no associated point, consider the commuting diagram

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "\times f"'] & \prod_{\mathfrak{p} \in \operatorname{Ass} M} M_{\mathfrak{p}} \arrow[d, "\times f"] \\ M \arrow[r, hook] & \prod_{\mathfrak{p} \in \operatorname{Ass} M} M_{\mathfrak{p}}, \end{tikzcd}$$

where the rows are the maps of (6.5.11.1). The vertical arrow on the right (multiplication by f) is an injection by hypothesis, so the vertical arrow on the left must be an injection as well. □

6.5.14. Associated points behave *fairly* well in exact sequences.

6.5.15. Proposition. — *Suppose*

(6.5.15.1) $$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence of A*-modules. Then*

(6.5.15.2) $$\operatorname{Ass} M' \subset \operatorname{Ass} M \subset \operatorname{Ass} M' \cup \operatorname{Ass} M''.$$

We come to our first complicated proof of the section.

Proof. The first inclusion of (6.5.15.2) was shown in Exercise 6.5.F.

Suppose next that $[\mathfrak{p}] \in \operatorname{Ass} M$, so there is some $m \in M$ with $\operatorname{Ann} m = \mathfrak{p}$. We wish to find a submodule of M' or M'' isomorphic to $A/\mathfrak{p}$. If this proposition were true, we would expect to find such a submodule in the "part of (6.5.15.1) spanned by m". So we consider instead the exact sequence

$$0 \longrightarrow Am \cap M' \longrightarrow Am \longrightarrow Am/(Am \cap M') \longrightarrow 0,$$

noting that the three modules appearing here are submodules of the corresponding modules in (6.5.15.1). So by Exercise 6.5.F it suffices to prove the result in this "special case", which can be rewritten as

$$0 \longrightarrow I/\mathfrak{p} \longrightarrow A/\mathfrak{p} \longrightarrow A/I \longrightarrow 0$$

where I is the annihilater of m considered as an element of the module $Am/(Am \cap M')$. For convenience, let $B = A/\mathfrak{p}$ (an integral domain), so we rewrite the exact sequence further as the top row of

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, hook] & B \arrow[r] \arrow[d, hook] & B/J \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \end{tikzcd}$$

Now localize the top row of B-modules at $(0) \subset B$, so it becomes an exact sequence of vector spaces over the fraction field $K(B)$, and the central element is one-dimensional:

$$0 \to J \otimes K(B) \to K(B) \to (B/J) \otimes K(B) \to 0.$$

Thus one of the outside terms $J \otimes K(B)$ and $(B/J) \otimes K(B)$ has a nonzero element, which (tracing our argument backwards) gives an element of M' or M'' whose annihilator is precisely $\mathfrak{p}$. $\square$

6.5.16. *Cautionary example.* The short exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

(and Easy Exercise 6.5.D) shows it is not always true that $\operatorname{Ass} M = \operatorname{Ass} M' \cup \operatorname{Ass} M''$. However, sometimes we can still ensure some associated primes of M'' lift to associated primes of M, as we shall see in §6.5.20.

6.5.17. Finitely generated modules over Noetherian rings have finitely many associated points/primes.

6.5.J. IMPORTANT EXERCISE. Suppose that M is a finitely generated module over a Noetherian ring A. Show that M has a (finite) filtration

(6.5.17.1)
$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M \quad \text{where } M_{i+1}/M_i \cong A/\mathfrak{p}_i \text{ for some prime } \mathfrak{p}_i.$$

Hint: Build (6.5.17.1) inductively from left to right, using Proposition 6.5.10, and show the process terminates.

6.5.K. EXERCISE. Suppose an A-module M has a finite filtration (6.5.17.1), with no assumptions of finite generation or Noetherianity. Show that every associated prime of M appears as one of the $\mathfrak{p}_i$. In particular, under this hypothesis (for example, if M is finitely generated over a Noetherian ring) M *has finitely many associated points/primes*. (Hint: Exercise 6.5.D and Proposition 6.5.15.)

6.5.18. *Caution: Non-associated prime ideals may unavoidably appear among the quotients in* (6.5.17.1). Example 6.5.16 shows that the non-associated prime ideals may be among the quotients in (6.5.17.1), although in that case it is because the filtration was chosen unwisely. But better choices will not always remedy the problem:

6.5.L. EXERCISE. Consider the module $M = (x, y) \subset A = k[x, y]$ over the ring A. Show that *any* filtration (6.5.17.1) of M contains a quotient $A/\mathfrak{p}_i$ where $\mathfrak{p}_i$ is *not* an associated prime.

6.5.19. *Remark.* However, not all is lost: Exercise 6.5.P will show that for any quotient $A/\mathfrak{p}_i$ in any filtration (6.5.17.1) of M, $\mathfrak{p}_i$ must *contain* an associated prime of M.

6.5.20. Associated points behave *fairly* well in exact sequences, continued.

6.5.21. Proposition. — *We continue to consider the short exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of A-modules. Suppose $\mathfrak{p} \in \operatorname{Ass} M''$, but $\mathfrak{p} \notin \operatorname{Supp} M'$ (a stronger hypothesis than $\mathfrak{p} \notin \operatorname{Ass} M'$). Then $\mathfrak{p} \in \operatorname{Ass} M$.

We come to our second complicated proof of the section.

Proof. Choose $m'' \in M''$ with $\mathfrak{p} = \operatorname{Ann} m''$ in M''. Choose a lift of $m \in M$ of $m'' \in M''$. We apply a strategy similar to that of our proof of Proposition 6.5.15. Consider the "inclusion of short exact sequences"

$$\begin{tikzcd} 0 \arrow[r] & \ker(Am \to Am'') \arrow[r] \arrow[d, hook] & Am \arrow[r] \arrow[d, hook] & Am'' \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0. \end{tikzcd}$$

As $\operatorname{Supp}(\ker(Am \to Am'')) \subset \operatorname{Supp} M'$, $\operatorname{Ass}(Am) \subset \operatorname{Ass} M$, and $\operatorname{Ass}(Am'') \subset \operatorname{Ass} M''$ (Exercises 6.5.B(a) and 6.5.F), we have reduced to considering the top row instead of the bottom row. The top row can be conveniently rewritten as

$$0 \longrightarrow \mathfrak{p}/I \longrightarrow A/I \longrightarrow A/\mathfrak{p} \longrightarrow 0$$

(here $I = \operatorname{Ann}(m)$) where our hypothesis translates to $[\mathfrak{p}] \notin \operatorname{Supp}(\mathfrak{p}/I)$. For convenience, let $B = A/I$ so we may now consider the sequence

$$0 \longrightarrow \mathfrak{q} \longrightarrow B \longrightarrow B/\mathfrak{q} \longrightarrow 0,$$

where $\mathfrak{q}$ is prime, with the confusion-inducing hypothesis $[\mathfrak{q}] \notin \operatorname{Supp} \mathfrak{q}$.

From the confusing hypothesis, there is an element b of B that vanishes on $\operatorname{Supp} \mathfrak{q}$ but doesn't vanish at $[\mathfrak{q}]$. Translation: (i) b lies in all the primes of $\operatorname{Supp} \mathfrak{q}$, but (ii) $b \notin \mathfrak{q}$. Then from (i) there is some power b^n of b that annihilates all elements of $\mathfrak{q}$ (Exercise 6.5.C). From (ii), $b^n \notin \mathfrak{q}$.

Then $\operatorname{Ann}(b^n)$ contains $\mathfrak{q}$ from (i), but does not contain any element of $B \setminus \mathfrak{q}$ from (ii), so $\operatorname{Ann}(b^n) = \mathfrak{q}$. hence $\mathfrak{q}$ is an associated prime of B, which (unwinding our argument) shows that $\mathfrak{p}$ is an associated prime of M. □

6.5.22. Minimal primes are associated.

6.5.M. EXERCISE: MINIMAL PRIMES ("IRREDUCIBLE COMPONENTS") ARE ASSOCIATED. Suppose M is a finitely generated module over Noetherian A, and $\mathfrak{p} \subset A$ is a prime ideal corresponding to an irreducible component of $\operatorname{Supp} M \subset \operatorname{Spec} A$. Show that $[\mathfrak{p}] \in \operatorname{Ass} M$. Hint: Exercise 6.5.J and Proposition 6.5.21.

6.5.23. *Non-Noetherian Remark.* By combining Proposition 6.5.13 with Exercise 6.5.M, we see that if A is a Noetherian ring, then any element of any minimal prime $\mathfrak{p}$ is a zerodivisor. This is true without Noetherian hypotheses: suppose $s \in \mathfrak{p}$. Then by minimality of $\mathfrak{p}$, $\mathfrak{p}A_{\mathfrak{p}}$ is the unique prime ideal in $A_{\mathfrak{p}}$, so the element $s/1$ of $A_{\mathfrak{p}}$ is nilpotent (because it is contained in all prime ideals of $A_{\mathfrak{p}}$, Theorem 3.2.13). Thus for some $t \in A \setminus \mathfrak{p}$, $ts^n = 0$, so s is a zerodivisor in A. We will use this in Exercise 12.1.G.

6.5.24. Proposition. — *Suppose M is a finitely generated module over a Noetherian ring A. Then $\operatorname{Supp} M = \overline{\operatorname{Ass} M}$.*

Proof. Combine Exercises 6.5.G and 6.5.M. □

6.5.N. EXERCISE. Suppose A is a Noetherian ring. Show that the locus of points $[\mathfrak{p}]$ where $A_{\mathfrak{p}}$ is nonreduced is the support of the nilradical $\operatorname{Supp} \mathfrak{N}$. Hence show that the "reduced locus" of a locally Noetherian scheme is open.

The following justifies a simple way of thinking about associated primes of a ring.

6.5.O. EXERCISE. Show that a prime ideal $\mathfrak{p} \subset A$ is an associated prime of A if and only if there is $f \in A$ such that $\operatorname{Supp} f = V(\mathfrak{p}) = \overline{[\mathfrak{p}]}$.

6.5.P. EXERCISE, PROMISED IN REMARK 6.5.19. Show that for each quotient in the filtration (6.5.17.1) of M, $\operatorname{Supp} A/\mathfrak{p}_i = \overline{[\mathfrak{p}]} \subset \operatorname{Supp} M$, and that every $\mathfrak{p}_i$ contains a minimal prime, and hence an associated prime.

6.5.25. "Support" and "associated points" commute with localization.

Suppose S is a multiplicative subset of A, and $\mathfrak{p} \subset A$ is a prime ideal not meeting S, so (abusing notation slightly) $[\mathfrak{p}] \in \operatorname{Spec} S^{-1}A \subset \operatorname{Spec} A$ (§3.2.9).

6.5.Q. EXERCISE (Supp COMMUTES WITH LOCALIZATION). Show that for any A-module M, $\operatorname{Supp}_A M \cap \operatorname{Spec} S^{-1}A = \operatorname{Supp}_{S^{-1}A} S^{-1}M$.

6.5.26. Proposition (Ass **commutes with localization). —** *For an* A*-module* M*, we have* $\operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A = \operatorname{Ass}_{S^{-1}A} S^{-1}M$.

Proof. We first show that $\operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A \subset \operatorname{Ass}_{S^{-1}A} S^{-1}M$. If $\mathfrak{p} \in \operatorname{Ass}_A M$, then we have an injection $A/\mathfrak{p} \hookrightarrow M$. Localizing by S (which preserves injectivity), we have $(S^{-1}A)/(S^{-1}\mathfrak{p}) \hookrightarrow M$. (Where did we use $\mathfrak{p} \in \operatorname{Spec} S^{-1}A$?)

We next show that $\operatorname{Ass}_{S^{-1}A} S^{-1}M \subset \operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A$. Suppose $\mathfrak{q} := S^{-1}\mathfrak{p} \in \operatorname{Ass}_{S^{-1}A} S^{-1}M$, so $\mathfrak{q} = \operatorname{Ann}_{S^{-1}A}(m/s)$, for some $s \in S$, and $m \in M$. Since the elements of S are units in $S^{-1}A$, we have that $\mathfrak{q} = \operatorname{Ann}_{S^{-1}A} m$. As support commutes with localization, $V(\mathfrak{p})$ must be an irreducible component of $\operatorname{Supp} m$ (as $\operatorname{Supp} m \cap \operatorname{Spec} S^{-1}M$ contains $[\mathfrak{q}]$, but no generizations of $[\mathfrak{q}]$). Then by Exercise 6.5.M, $\mathfrak{p}$ is an associated prime of M. □

6.5.27. Embedded points/primes.

6.5.28. *Definition.* The associated points that are *not* the generic points of irreducible components of $\operatorname{Supp} M$ are called **embedded points**, and their corresponding primes are called **embedded primes**. For example, the scheme of Figure 6.1 has three embedded primes.

6.5.29. *Remark.* Exercise 6.5.E translates to "hypersurfaces in $\mathbb{A}^n_k$ have no embedded points". More generally, if A is a unique factorization domain, then $\operatorname{Spec} A/(f)$ has no embedded points for any $f \in A$. Generalizing in a different direction, we will see that "complete intersections have no embedded points" in §29.2.7.

6.5.R. EXERCISE. Suppose A is a reduced ring (i.e., A has no nonzero nilpotents). Show that $\operatorname{Spec} A$ has no embedded primes. (Hints: if $\mathfrak{p} = \operatorname{Ann} a$ is an embedded prime, show that you can find an element b of $\mathfrak{p}$ not contained in any of the minimal primes of A. From $ab = 0$, show that a is contained in all the minimal primes. Show that a vanishes at all points of $\operatorname{Spec} A$, and hence by Theorem 3.2.13 is nilpotent.)

Thus *reduced rings have no embedded primes.* Even better: *the only elements of a ring that an embedded prime can annihilate are nilpotents.*

6.5.30. *Remark.* The converse to Exercise 6.5.R is false. Rings without embedded primes can still have nilpotents — witness $k[x]/(x^2)$.

6.5.S. EXERCISE. Show that if $\mathfrak{p}$ is an embedded prime of a ring A, then $A_{\mathfrak{p}}$ is nonreduced.

6.5.31. Get your hands dirty: Explicit algebraic exercises.

6.5.T. EXERCISE. (See Figure 4.4.) Suppose f is a function on $\operatorname{Spec} k[x,y]/(y^2, xy)$, i.e., $f \in k[x,y]/(y^2, xy)$. Show that $\operatorname{Supp} f$ is either the empty set, or the origin, or the entire space. Hence find the associated points of $\operatorname{Spec} k[x,y]/(y^2, xy)$.

6.5.U. EXERCISE (CONTINUING THE PREVIOUS EXERCISE). Show explicitly by hand that $f \in k[x,y]/(y^2, xy)$ is a zerodivisor if and only if $f(0,0) = 0$.

6.5.V. EXERCISE (PRACTICE WITH FUZZY PICTURES). Suppose $X = \operatorname{Spec} \mathbb{C}[x,y]/I$, and that the associated points of X are $[(y-x^2)]$, $[(x-1, y-1)]$, and $[(x-2, y-2)]$.
(a) Sketch X as a subset of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]$, including fuzz.
(b) Do you have enough information to know if X is reduced?
(c) Do you have enough information to know if $x+y-2$ is a zerodivisor? How about $x+y-3$? How about $y-x^2$? (Exercise 6.5.W will verify that such an X actually exists.)

6.5.W. EXERCISE. Let $I = (y-x^2)^3 \cap (x-1, y-1)^{15} \cap (x-2, y-2)$. Show that $X = \operatorname{Spec} \mathbb{C}[x,y]/I$ satisfies the hypotheses of Exercise 6.5.V. (Rhetorical question: Is there a "smaller" example? Is there a "smallest"?)

6.5.32. Geometric definitions.

6.5.X. EXERCISE/DEFINITION. Define the **associated points of a locally Noetherian scheme**. (Idea/hint: do it affine-locally.)

6.5.Y. EXERCISE. Suppose X is a locally Noetherian scheme, and $U \subset X$ is an open subscheme. Show that the natural map

$$\Gamma(U, \mathscr{O}_X) \longrightarrow \prod_{p \in \operatorname{Ass} X \cap U} \mathscr{O}_{X,p} \tag{6.5.32.1}$$

(cf. (6.5.11.1)) is an injection.

6.5.33. Generalizing the fraction field: the total fraction ring.

6.5.34. *Definitions: Rational functions on locally Noetherian schemes.* A **rational function** on a locally Noetherian scheme is an element of the image of $\Gamma(U, \mathscr{O}_U)$ in (6.5.32.1) for some U containing all the associated points. Equivalently, the set of rational functions is the colimit of $\mathscr{O}_X(U)$ over all open sets containing the associated points.

For example, on $\operatorname{Spec} k[x,y]/(y^2, xy)$ (Figure 4.4), $\frac{x-2}{(x-1)(x-3)}$ is a rational function, but $\frac{x-2}{x(x-1)}$ is not.

A rational function has a maximal **domain of definition**, because any two actual functions on an open set (i.e., sections of the structure sheaf over that open set) that agree as "rational functions" (i.e., on small enough open sets containing associated points) must be the same function, by the injectivity of (6.5.32.1). We say that a rational function f is **regular at a point** p if p is contained in this maximal domain of definition (or equivalently, if there is some open set containing p where f is defined). For example, on $\operatorname{Spec} k[x,y]/(y^2, xy)$, the rational function $\frac{x-2}{(x-1)(x-3)}$ has domain of definition consisting of everything but 1 and 3 (i.e., $[(x-1)]$ and $[(x-3)]$), and is regular away from those two points. A rational function is **regular** if it is regular at all points. (Unfortunately, "regular" is a regularly overused word in mathematics, and in algebraic geometry in particular.)

The complement of the domain of definition of a rational function f is called the **indeterminacy locus of** f (a phrase we'll see again in §11.4.3).

The rational functions form a ring, called the **total fraction ring** or **total quotient ring** of X. If $X = \operatorname{Spec} A$ is affine, then this ring is called the **total fraction (or quotient) ring** of A. If X is integral, the total fraction ring is the function field $K(X)$ — the stalk at the generic point — so this extends our earlier Definition 5.2.I of $K(\cdot)$.

6.5.Z. EXERCISE. Show that the ring of rational functions on a locally Noetherian scheme is the colimit of the functions over all open sets containing the associated points:

$$\operatorname{colim}_{U : \operatorname{Ass} X \subset U} \mathscr{O}(U).$$

Slightly better (but slightly different): show that a rational function is the data of a function f defined on an open set U containing all associated points, where two such data (U, f) and (U', f') define the same rational function if and only if $f_{U \cap U'} = f'_{U \cap U'}$ (cf. 1.4.9). If X is reduced, show that this is the same as requiring that they are defined on an open set of each of the irreducible components.

6.5.35. *Remark: Associated points of integral schemes.* In order for some of our discussion elsewhere to make sense in non-Noetherian settings, we note that the notion of associated points for integral schemes works perfectly, because it works for integral domains — only the generic point is associated. In particular, the definition above of rational functions on an integral scheme X agrees with Definition 5.2.I, as precisely elements of the function field $K(X)$.

6.5.36. $\star\star$ **Aside: Primary ideals and associated primes.** Primary decomposition was introduced by the world chess champion E. Lasker, and axiomatized by world math champion E. Noether. We won't need it, but here is the beginning of the story, for the curious reader. An ideal $I \subset A$ in a ring is **primary** if $I \neq A$, and $xy \in I$ implies either $x \in I$ or $y^n \in I$ for some $n > 0$. In this case, $\sqrt{I}$ is prime, and I is said to be $\mathfrak{p}$-primary. Equivalently, if $I \subset A$ then I is $\mathfrak{p}$-primary if and only if A/I has only one associated prime $\mathfrak{p}$. If I is an ideal of a Noetherian ring A, then the associated prime ideals A/I turn out to be precisely the radicals of ideals in a primary decomposition. See [**E**, §3.3], for example, for more of this important story.

6.6 $\star\star$ Coherent modules over non-Noetherian rings

This section is intended for people who might work with non-Noetherian rings, or who otherwise might want to understand coherent sheaves in a more general