CHAPTER 28

Cohomology and base change theorems

28.1 Statements and applications

Higher pushforwards are easy to define, but it is hard to get a geometric sense of what they are, or how they behave. For example, given a morphism $\pi \colon X \to Y$, and a quasicoherent sheaf $\mathscr{F}$ on X, you might reasonably hope that the fibers of $R^i\pi_*\mathscr{F}$ are the cohomologies of $\mathscr{F}$ along the fibers. More precisely, given $\psi \colon q \to Y$ corresponding to the inclusion of a point (better: $\psi \colon \operatorname{Spec} \kappa(q) \to Y$), yielding the fibered diagram

(28.1.0.1)
$$\begin{array}{ccc} X_q & \xrightarrow{\psi'} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ q & \xrightarrow{\psi} & Y, \end{array}$$

one might hope that the morphism

$$\boxed{\phi_q^p \colon \psi^*(R^p\pi_*\mathscr{F}) \longrightarrow H^p(X_q, \mathscr{F}|_{X_q})}$$

(given in Exercise 19.8.C) is an isomorphism. (Note: $\mathscr{F}|_{X_q}$ and $(\psi')^*\mathscr{F}$ are symbols for the same thing. The first is often preferred, but we sometimes use the second because we will consider more general ψ and ψ'.) We could then picture $R^i\pi_*\mathscr{F}$ as somehow fitting together the cohomology groups of fibers into a coherent sheaf. (Warning: this is too much to hope for, see Exercise 28.1.A.)

It would also be nice if $h^p(X_q, (\psi')^*\mathscr{F})$ was constant, and ϕ_q^p put them together into a nice locally free sheaf (vector bundle) $\pi_*\mathscr{F}$.

The base change $\psi \colon q \to Y$ should not be special. As long as we are dreaming, we may as well hope that in good circumstances, given a Cartesian diagram (19.8.4.1)

(28.1.0.2)
$$\begin{array}{ccc} W & \xrightarrow{\psi'} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\psi} & Y, \end{array}$$

the natural morphism

(28.1.0.3)
$$\boxed{\phi_Z^p \colon \psi^*(R^p\pi_*\mathscr{F}) \to R^p\pi'_*(\psi')^*\mathscr{F}}$$

of sheaves on Z (Exercise 19.8.B(a)) is an isomorphism. (In some cases, we can already address this question. For example, cohomology commutes with flat base change, Theorem 25.2.8, so the result holds if ψ is flat.

We formalize our dreams into three nice properties that we might wish in this situation. We will see that they are closely related. Suppose $\mathscr{F}$ is a coherent sheaf on X, $\pi\colon X \to Y$ is proper, Y (hence X) is Noetherian, and $\mathscr{F}$ is flat over Y.

(a) Given a Cartesian square (28.1.0.1), is $\phi^p_q\colon R^p\pi_*\mathscr{F}\otimes\kappa(q) \to H^p(X_q, \mathscr{F}|_{X_q})$ an isomorphism?
(b) Given a Cartesian square (28.1.0.2), is $\phi^p_Z\colon \psi^*(R^p\pi_*\mathscr{F}) \to R^p\pi'_*(\psi')^*\mathscr{F}$ an isomorphism?
(c) Is $R^p\pi_*\mathscr{F}$ locally free?

We turn first to property (a). The dimension of the left side $R^p\pi_*\mathscr{F}\otimes\kappa(q)$ is an upper semicontinuous function of $q \in Y$ by upper semicontinuity of rank of finite type quasicoherent sheaves (Exercise 14.4.J). The Semicontinuity Theorem states that the dimension of the right is also upper semicontinuous. More formally:

28.1.1. Semicontinuity Theorem. — *Suppose $\pi\colon X \to Y$ is a proper morphism of Noetherian schemes, and $\mathscr{F}$ is a coherent sheaf on X flat over Y. Then for each $p \geq 0$, the function $Y \to \mathbb{Z}$ given by $q \mapsto \dim_{\kappa(q)} H^p(X_q, \mathscr{F}|_{X_q})$ is an upper semicontinuous function of $q \in Y$.*

Translation: ranks of cohomology groups are upper semicontinuous in proper flat families. (A proof will be given in §28.2.4.)

28.1.2. *Example.* You may already have seen an example of cohomology groups jumping, in §25.4.14. Here is a simpler example, albeit not of the structure sheaf. Let (E, p_0) be an elliptic curve over a field k, and consider the projection $\pi\colon E \times E \to E$ to the second factor. Let $\mathscr{L}$ be the invertible sheaf (line bundle) on $E \times E$ corresponding to the divisor that is the diagonal, minus the section of $p_0 \times E$ of π (where $p_0 \in E$). Then $\mathscr{L}|_{p_0}$ (i.e., $\mathscr{L}|_{E\times p_0}$) is trivial, but $\mathscr{L}|_p$ is non-trivial for any $p \neq p_0$ (as we showed in our study of genus 1 curves, in §20.9). Thus $h^0(E, \mathscr{L}|_p)$ is 0 in general, but jumps to 1 for $p = p_0$.

28.1.A. EXERCISE. Show that $\pi_*\mathscr{L} = 0$. Thus we cannot picture $\pi_*\mathscr{L}$ as "gluing together" h^0 of the fibers; in this example, cohomology does not commute with "base change" or "taking fibers".

28.1.3. *Side Remark.* In characteristic 0, the cohomology of $\mathscr{O}$ doesn't jump in smooth families. Over $\mathbb{C}$, this is because Betti numbers are constant in connected families, and (22.5.11.1) (from Hodge theory) expresses the Betti constants h^k_{Betti} as sums (over $i + j = k$) of upper semicontinuous functions $h^j(\Omega^i)$, so the Hodge numbers $h^j(\Omega^i)$ must in fact be constant. The general characteristic 0 case can be reduced to $\mathbb{C}$ by an application of the Lefschetz principle (which also arose in §22.5.9). But ranks of cohomology groups of $\mathscr{O}$ for smooth families of varieties *can* jump in positive characteristic (see for example [**MO70920**]). Also, the example of §25.4.14 shows that the "smoothness" hypothesis cannot be removed.

28.1.4. Grauert's Theorem. If $R^p\pi_*\mathscr{F}$ is locally free (property (c)) and ϕ^p_q is an isomorphism (property (a)), then $h^p(X_q, \mathscr{F}|_{X_q})$ is clearly locally constant. The following is a partial converse.

28.1.5. Grauert's Theorem. — *If $\pi\colon X \to Y$ is proper, Y is reduced and locally Noetherian, $\mathscr{F}$ is a coherent sheaf on X flat over Y, and $h^p(X_q, \mathscr{F}|_{X_q})$ is a locally constant function of $q \in Y$, then $R^p\pi_*\mathscr{F}$ is locally free, and ϕ^p_Z is an isomorphism for all $\psi\colon Z \to Y$.*

In other words, if cohomology groups of fibers have locally constant dimension (over a reduced base), then they can be fit together to form a vector bundle, and the fiber of the pushforward is identified with the cohomology of the fiber. Our dreams at the start of this chapter have come true!

(See §28.2.12 to remove Noetherian hypotheses.)

We further note that if Y is integral, π is proper, and $\mathscr{F}$ is a coherent sheaf on X flat over Y, then by the Semicontinuity Theorem 28.1.1 there is a dense open subset of Y on which $R^p\pi_*\mathscr{F}$ is locally free (and on which the fiber of the pth higher pushforward is the pth cohomology of the fiber).

The following statement is even more magical than Grauert's Theorem 28.1.5.

28.1.6. Cohomology and Base Change Theorem. — *Suppose π is proper, Y is locally Noetherian, $\mathscr{F}$ is coherent over X and flat over Y, and ϕ^p_q is surjective. Then the following hold.*

(i) *There is an open neighborhood U of q such that for any $\psi\colon Z \to U$, ϕ^p_Z is an isomorphism. In particular, ϕ^p_q is an isomorphism.*
(ii) *Furthermore,*
   (a) *ϕ^{p-1}_q is surjective (hence an isomorphism by (i)) if and only if*
   (b) *$R^p\pi_*\mathscr{F}$ is locally free in some open neighborhood of q (or equivalently, $(R^p\pi_*\mathscr{F})_q$ is a free $\mathscr{O}_{Y,q}$-module, Exercise 14.4.F).*
   (These then imply that $h^p(X_r, \mathscr{F}|_{X_r})$ is constant for r in an open neighborhood of q.)

(Proofs of Theorems 28.1.5 and 28.1.6 will be given in §28.2. Note in (ii) that if $p = 0$, ϕ^{p-1}_q is automatically surjective, as ϕ^{-1}_q is the zero map. See §28.2.12 to remove Noetherian hypotheses.)

This is amazing: the hypothesis that ϕ^p_q is surjective involves what happens only over points q of X, with *reduced* structure, yet it has implications over the (possibly nonreduced) scheme as a whole! This might remind you of the local criterion for flatness (Theorem 25.6.2), which indeed is the key technical ingredient of the proof.

Here are some consequences.

28.1.B. EXERCISE. Use Theorem 28.1.6 to give a second solution to Exercise 25.4.E. (This is a big weapon to bring to bear on this problem, but it is still enlightening; your original solution to Exercise 25.4.E may foreshadow the proof of the Cohomology and Base Change Theorem 28.1.6.)

28.1.C. EXERCISE. Suppose $\pi\colon X \to Y$ is proper, Y is locally Noetherian, and $\mathscr{F}$ is a coherent sheaf on X, flat over Y. Suppose further that $H^p(X_q, \mathscr{F}|_{X_q}) = 0$ for some $q \in Y$. Show that there is an open neighborhood U of q such that $(R^p\pi_*\mathscr{F})|_U = 0$.

28.1.D. EXERCISE. Suppose $\pi\colon X \to Y$ is proper, Y is locally Noetherian, and $\mathscr{F}$ is a coherent sheaf on X, flat over Y. Suppose further that $H^p(X_q, \mathscr{F}|_{X_q}) = 0$ for all

$q \in Y$. Show that the $(p-1)$st cohomology commutes with arbitrary base change: ϕ_Z^{p-1} is an isomorphism for all $\psi\colon Z \to Y$.

28.1.E. EXERCISE. Suppose π is proper, Y is locally Noetherian, and $\mathscr{F}$ is a coherent sheaf on X flat over Y. Suppose further that $R^p\pi_*\mathscr{F} = 0$ for $p \geq p_0$. Show that $H^p(X_q, \mathscr{F}|_{X_q}) = 0$ for all $q \in Y$, $p \geq p_0$.

28.1.F. EXERCISE. Suppose π is proper, Y is locally Noetherian, and $\mathscr{F}$ is a coherent sheaf on X, flat over Y. Suppose further that Y is reduced. Show that there exists a dense open subset U of Y such that ϕ_Z^p is an isomorphism for all $\psi\colon Z \to U$ and all p. (Hint: find suitable open neighborhoods of the generic points of Y. See Exercise 25.2.M and the paragraph following it.)

28.1.7. An important class of morphisms: Proper, $\mathscr{O}$-connected morphisms $\pi\colon X \to Y$ of locally Noetherian schemes.

If a morphism $\pi\colon X \to Y$ satisfies the property that the natural map $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$ is an isomorphism, we say that π is **$\mathscr{O}$-connected**. Clearly the notion of $\mathscr{O}$-connectedness is local on the target, and preserved by composition.

28.1.G. EASY EXERCISE. Show that proper $\mathscr{O}$-connected morphisms of locally Noetherian schemes are surjective.

28.1.8. We will soon meet Zariski's Connectedness Lemma 30.5.1, which shows that proper, $\mathscr{O}$-connected morphisms of locally Noetherian schemes have connected fibers. In some sense, this class of morphisms is really the right class of morphisms capturing what we might want by "connected fibers"; this is the motivation for the terminology. The following result gives some evidence for this point of view, in the flat context.

28.1.H. IMPORTANT EXERCISE. Suppose $\pi\colon X \to Y$ is a proper *flat* morphism of locally Noetherian schemes, whose fibers satisfy $h^0(X_q, \mathscr{O}_{X_q}) = 1$. (Important remark: this is satisfied if π has geometrically connected and geometrically reduced fibers, by §11.5.7.) Show that π is $\mathscr{O}$-connected. Hint: consider

$$\mathscr{O}_Y \otimes \kappa(q) \longrightarrow (\pi_*\mathscr{O}_X) \otimes \kappa(q) \xrightarrow{\phi_q^0} H^0(X_q, \mathscr{O}_{X_q}) \cong \kappa(q)\,.$$

The composition is surjective, hence ϕ_q^0 is surjective, hence it is an isomorphism by the Cohomology and Base Change Theorem 28.1.6(i). Then by the Cohomology and Base Change Theorem 28.1.6(ii), $\pi_*\mathscr{O}_X$ is locally free, thus of rank 1. Perhaps use Nakayama's Lemma to show that a map of invertible sheaves $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$ that is an isomorphism on fibers over points (with reduced structure) is necessarily an isomorphism of sheaves.

28.1.9. $\star$ *Unimportant remark.* This class of proper, $\mathscr{O}$-connected morphisms is not preserved by arbitrary base change, and thus is not "reasonable" in the sense of §8.1. But you can show that they are preserved by *flat* base change, using the fact that cohomology commutes with flat base change, Theorem 25.2.8. Furthermore, the conditions of Exercise 28.1.H behave well under base change, and Noetherian hypotheses can be removed from the Cohomology and Base Change Theorem 28.1.6 (at the expense of finitely presented hypotheses, see §28.2.12), so the class of morphisms $\pi: X \to Y$ that are proper, finitely presented, and flat, with geometrically connected and geometrically reduced fibers, *is* "reasonable" (and useful).

28.1.10. We next address the following question. Suppose $\pi: X \to Y$ is a morphism of schemes. Given an invertible sheaf $\mathscr{L}$ on X, we ask when it is the pullback of an invertible sheaf $\mathscr{M}$ on Y. For this to be true, we certainly need that $\mathscr{L}$ is trivial on the fibers. We will see that if π is a proper $\mathscr{O}$-connected morphism of locally Noetherian schemes, then this often suffices. Given $\mathscr{L}$, we recover $\mathscr{M}$ as $\pi_*\mathscr{L}$; the fibers of $\mathscr{M}$ are one-dimensional, and glue together to form a line bundle. We now begin to make this precise.

28.1.I. EXERCISE. Suppose $\pi: X \to Y$ is a proper, $\mathscr{O}$-connected morphism of locally Noetherian schemes. Show that if $\mathscr{M}$ is any invertible sheaf on Y, then the natural morphism $\mathscr{M} \to \pi_*\pi^*\mathscr{M}$ is an isomorphism. In particular, we can recover $\mathscr{M}$ from $\pi^*\mathscr{M}$ by applying the pushforward π_*.

28.1.11. Proposition. — *Suppose $\pi: X \to Y$ is a* **flat**, *proper morphism of locally Noetherian schemes with geometrically connected and geometrically reduced fibers (hence $\mathscr{O}$-connected, by Exercise 28.1.H). Suppose also that Y is reduced, and $\mathscr{L}$ is an invertible sheaf on X that is trivial on the fibers of π (i.e., $\mathscr{L}|_{X_q}$ is a trivial invertible sheaf on X_q for all $q \in Y$). Then $\pi_*\mathscr{L}$ is an invertible sheaf on Y (call it $\mathscr{M}$), and the natural map $\pi^*\mathscr{M} \to \mathscr{L}$ is an isomorphism.*

Proof. By Grauert's Theorem 28.1.5, $\pi_*\mathscr{L}$ is locally free of rank 1 (again, call it $\mathscr{M}$), and $\mathscr{M} \otimes_{\mathscr{O}_Y} \kappa(q) \to H^0(X_q, \mathscr{L}|_{X_q})$ is an isomorphism. We have a natural map of invertible sheaves $\pi^*\mathscr{M} = \pi^*\pi_*\mathscr{L} \to \mathscr{L}$. To show that it is an isomorphism, we need only show that it is surjective. (Do you see why? If A is a ring, and $\phi: A \to A$ is a surjection of A-modules, why is ϕ an isomorphism?) For this, it suffices to show that it is surjective on the fibers of π. (Do you see why? Hint: if the cokernel of the map is not 0, then it is not 0 above some point of Y.) But this follows from the first line of the proof (using for example that $\mathscr{M} \cong \mathscr{O}$ in a neighborhood of q). □

Proposition 28.1.11 has some pleasant consequences. For example, if you have two invertible sheaves $\mathscr{A}$ and $\mathscr{B}$ on X that are isomorphic on every fiber of π, then they differ by a pullback of an invertible sheaf on Y: just apply Proposition 28.1.11 to $\mathscr{A} \otimes \mathscr{B}^\vee$.

28.1.12. Projective bundles.

28.1.J. EXERCISE. Let X be a locally Noetherian scheme, and let $\mathrm{pr}_1: X \times \mathbb{P}^n \to X$ be the projection onto the first factor. Suppose $\mathscr{L}$ is a line bundle on $X \times \mathbb{P}^n$, whose degree on every fiber of pr_1 is zero. Use the Cohomology and Base Change Theorem 28.1.6 to show that $(\mathrm{pr}_1)_*\mathscr{L}$ is an invertible sheaf on X. Show that the natural map $\mathrm{pr}_1^*((\mathrm{pr}_1)_*\mathscr{L}) \to \mathscr{L}$ is an isomorphism, as at the end of proof of Proposition 28.1.11.

Your argument will apply just as well to the situation where $\mathrm{pr}_1: X \times \mathbb{P}^n \to X$ is replaced by a $\mathbb{P}^n$-bundle over X, $\mathrm{pr}_1: Z \to X$; or by $\mathrm{pr}_1: Z \to X$ which is a proper smooth morphism whose geometric fibers are integral curves of genus 0.

Furthermore, the locally Noetherian hypotheses can be removed, see §28.2.12.

28.1.K. EXERCISE. Suppose X is a connected Noetherian scheme. Show that $\operatorname{Pic}(X \times \mathbb{P}^n) \cong \operatorname{Pic} X \times \mathbb{Z}$. Hint: the map $\operatorname{Pic} X \times \operatorname{Pic} \mathbb{P}^n \to \operatorname{Pic}(X \times \mathbb{P}^n)$ is given by $(\mathscr{L}, \mathscr{O}(m)) \mapsto \mathrm{pr}_1^* \mathscr{L} \otimes \mathrm{pr}_2^* \mathscr{O}(m)$, where $\mathrm{pr}_1 \colon X \times \mathbb{P}^n \to X$ and $\mathrm{pr}_2 \colon X \times \mathbb{P}^n \to \mathbb{P}^n$ are the projections from $X \times \mathbb{P}^n$ to its factors. (The notation $\boxtimes$ is often used for this construction, see §17.4.9.)

A very similar argument will show that if Z is a $\mathbb{P}^n$-bundle over X, then $\operatorname{Pic} Z \cong \operatorname{Pic} X \times \mathbb{Z}$. You will undoubtedly also be able to figure out the right statement if X is not connected.

28.1.13. *Remark.* As mentioned in §20.10.1, the Picard group of a scheme often "wants to be a scheme". You may be able to make this precise in the case of $\operatorname{Pic} \mathbb{P}^n_{\mathbb{Z}}$. In this case, the *scheme* $\operatorname{Pic} \mathbb{P}^n_{\mathbb{Z}}$ is "$\mathbb{Z}$ copies of $\operatorname{Spec} \mathbb{Z}$", with the "obvious" group scheme structure. Can you figure out what functor it represents? Can you show that it represents this functor? This will require extending Exercise 28.1.K out of the Noetherian setting, using §28.2.12.

28.1.L. EXERCISE. Suppose $\pi \colon X \to Y$ is a projective flat morphism over a Noetherian integral scheme, all of whose geometric fibers are isomorphic to $\mathbb{P}^n$ (over the appropriate field). Show that π is a projective bundle if and only if there is an invertible sheaf $\mathscr{L}$ on X that restricts to $\mathscr{O}(1)$ on all the geometric fibers. (One direction is clear: if it is a projective bundle, then it has a $\mathscr{O}(1)$ which comes from the projectivization, see Exercise 18.2.D. In the other direction, the candidate vector bundle is $\pi_* \mathscr{L}$. Show that it is indeed a locally free sheaf of the desired rank. Show that its projectivization is indeed $\pi \colon X \to Y$.)

Caution: the map $\pi \colon \operatorname{Proj} \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2) \to \operatorname{Spec} \mathbb{R}$ shows that not every projective flat morphism over a Noetherian integral scheme, all of whose geometric fibers are isomorphic to $\mathbb{P}^n$, is necessarily a $\mathbb{P}^n$-bundle. However, *Tsen's Theorem* implies that if the target is a *smooth curve over an algebraically closed field*, then the morphism *is* a $\mathbb{P}^n$-bundle (see [**GS**, Thm. 6.2.8]). Example 19.4.7 shows that "curve" cannot be replaced by "5-fold" in this statement — the "universal smooth plane conic" is not a $\mathbb{P}^1$-bundle over the parameter space $U \subset \mathbb{P}^5$ of smooth plane conics. If you wish, you can extend Example 19.4.7 to show that "curve" cannot even be replaced by "surface". (Just replace the $\mathbb{P}^5$ of all conics with a generally chosen $\mathbb{P}^2$ of conics — but then figure out what goes wrong if you try to replace it with a generally chosen $\mathbb{P}^1$ of conics.)

28.1.M. EXERCISE. Suppose $\pi \colon X \to Y$ is the projectivization of a vector bundle $\mathscr{F}$ over a locally Noetherian scheme (i.e., $X \cong \mathcal{P}roj \operatorname{Sym}^\bullet \mathscr{F}$). Recall from §18.2.3 that for any invertible sheaf $\mathscr{L}$ on Y, $X \cong \mathcal{P}roj \operatorname{Sym}^\bullet (\mathscr{F} \otimes \mathscr{L})$). Show that these are the only ways in which it is the projectivization of a vector bundle. (Hint: recover $\mathscr{F}$ by pushing forward $\mathscr{O}(1)$.)

28.1.14. The Hodge bundle.

28.1.N. EXERCISE (THE HODGE BUNDLE). Suppose $\pi \colon X \to Y$ is a flat proper morphism of locally Noetherian schemes, and the fibers of π are regular irreducible curves of genus g. (By Theorem 26.2.2 it is a smooth morphism of relative dimension 1, and $\Omega_{X/Y}$ is a line bundle.) Show that $\pi_*\Omega_{X/Y}$ is a locally free sheaf on Y of rank g, and that the construction of π commutes with base change: given a Cartesian square

(28.1.14.1)
$$\begin{array}{ccc} X' & \xrightarrow{\psi'} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{\psi} & Y, \end{array}$$

there is a canonical isomorphism

$$\left(\pi'_*\Omega_{X'/Y'}\right) \xleftrightarrow{\sim} \psi^*\left(\pi_*\Omega_{X/Y}\right).$$

(The locally free sheaf $\pi_*\Omega_{X/Y}$ is called the **Hodge bundle**.) Hint: use the Cohomology and Base Change Theorem 28.1.6 twice, once with $p = 2$, and once with $p = 1$.

28.2 ⋆ Proofs of cohomology and base change theorems

The key to proving the Semicontinuity Theorem 28.1.1, Grauert's Theorem 28.1.5, and the Cohomology and Base Change Theorem 28.1.6 is the following wonderful idea of Mumford (see [**Mu3**, p. 47 Lem. 1]). It turns questions of pushforwards (and how they behave under arbitrary base change) into something computable with vector bundles (hence questions of linear algebra). After stating it, we will interpret it.

28.2.1. Key Theorem. — *Suppose* $\pi\colon X \to \operatorname{Spec} B$ *is a proper morphism, and* $\mathscr{F}$ *is a coherent sheaf on* X*, flat over* $\operatorname{Spec} B$*, where* B *is Noetherian. Then there is a complex*

(28.2.1.1)
$$\cdots \xrightarrow{\delta^{-2}} K^{-1} \xrightarrow{\delta^{-1}} K^0 \xrightarrow{\delta^0} K^1 \xrightarrow{\delta^1} \cdots \longrightarrow K^n \xrightarrow{\delta^r} 0$$

of finitely generated free B*-modules and an isomorphism of functors*

(28.2.1.2)
$$H^p(X \times_B A, \mathscr{F} \otimes_B A) \xleftrightarrow{\sim} H^p(K^\bullet \otimes_B A)$$

for all p*, for all ring maps* $B \to A$*. (Here* A *needn't be Noetherian.)*

Because (28.2.1.1) is a complex of *free* B-modules, all of the information is contained in the maps, which are matrices with entries in B. This will turn questions about cohomology (and base change) into questions about linear algebra. For example, semicontinuity will turn into the fact that ranks of matrices (with functions as entries) drop on closed subsets (§12.4.4(ii)).

Although the complex (28.2.1.1) is infinite, by (28.2.1.2) it has no cohomology in negative degree, even after any ring extension $B \to A$ (as the left side of (28.2.1.2) is 0 for $p < 0$).

The idea behind the proof is as follows: take the Čech complex, produce a complex of finite rank free modules mapping to it "with the same cohomology" (a *quasiisomorphic complex*, §19.2.3). We first construct the complex so that (28.2.1.2) holds for $B = A$ in the next lemma, and then show the same complex works for general A, in Lemma 28.2.3 immediately thereafter.

28.2.2. Lemma. — *Let* B *be a Noetherian ring. Suppose* $C^\bullet$ *is a complex of* B*-modules such that* $H^i(C^\bullet)$ *are finitely generated* B*-modules, and such that* $C^p = 0$ *for* $p > n$*. Then there exists a complex* $K^\bullet$ *of finite rank free* B*-modules such that* $K^p = 0$ *for* $p > n$*, and a homomorphism of complexes* $\alpha \colon K^\bullet \to C^\bullet$ *such that* α *induces isomorphisms* $H^i(K^\bullet) \xrightarrow{\sim} H^i(C^\bullet)$ *for all* i*.*

Proof. We build this complex inductively. (This may remind you of Hint 24.3.3.) Assume we have defined $(K^p, \alpha^p, \delta^p)$ for $p \geq m+1$ (as in (28.2.2.1)) such that the squares commute, and the top row is a complex, and α^q defines an isomorphism of cohomology $H^q(K^\bullet) \to H^q(C^\bullet)$ for $q \geq m+2$ and a surjection $\ker(\delta^{m+1}) \to H^{m+1}(C^\bullet)$, and the K^p are finite rank free B-modules. (Our base case is $m = p$: take $K^n = 0$ for $n > p$.)

(28.2.2.1)
$$\begin{array}{ccccccccc} & & & & K^{m+1} & \xrightarrow{\delta^{m+1}} & K^{m+2} & \xrightarrow{\delta^{m+2}} & \cdots \\ & & & & \downarrow{\scriptstyle \alpha^{m+1}} & & \downarrow{\scriptstyle \alpha^{m+2}} & & \\ \cdots \longrightarrow C^{m-1} & \xrightarrow[\epsilon^{m-1}]{} & C^m & \xrightarrow[\epsilon^m]{} & C^{m+1} & \xrightarrow[\epsilon^{m+1}]{} & C^{m+2} & \longrightarrow & \cdots . \end{array}$$

We construct $(K^m, \delta^m, \alpha^m)$. Choose generators of $H^m(C^\bullet)$, say $c_1, \dots, c_M$. Let
$$D^{m+1} := \ker\left(\ker(\delta^{m+1}) \stackrel{\alpha^{m+1}}{\longrightarrow} H^{m+1}(C^\bullet)\right).$$

Choose generators of D^{m+1}, say $d_1, \dots, d_N$. (This is where we use the Noetherian hypotheses — to ensure this kernel D^{m+1} is finitely generated.) Let $K^m = B^{\oplus(M+N)}$. Define $\delta^m \colon K^m \to K^{m+1}$ by sending the last N generators to $d_1, \dots, d_N$, and the first M generators to 0. Define α^m by sending the first M generators of $B^{\oplus(M+N)}$ to (lifts of) $c_1, \dots, c_M$, and sending the last N generators to arbitrarily chosen lifts of the $\alpha^{m+1}(d_i)$ (as the $\alpha^{m+1}(d_i)$ are 0 in $H^{m+1}(C^\bullet)$, and thus lie in the image of ϵ^m), so the square (with upper left corner K^m) commutes. Then by construction, we have completed our inductive step:

$$\begin{array}{ccccccccc} & & K^m & \xrightarrow{\delta^m} & K^{m+1} & \xrightarrow{\delta^{m+1}} & K^{m+2} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \alpha^m} & & \downarrow{\scriptstyle \alpha^{m+1}} & & \downarrow{\scriptstyle \alpha^{m+2}} & & \\ \cdots \longrightarrow C^{m-1} & \xrightarrow[\epsilon^{m-1}]{} & C^m & \xrightarrow[\epsilon^m]{} & C^{m+1} & \xrightarrow[\epsilon^{m+1}]{} & C^{m+2} & \longrightarrow & \cdots . \end{array}$$

□

28.2.3. Lemma. — *Suppose* $\alpha \colon K^\bullet \to C^\bullet$ *is a morphism of complexes of* **flat** B*-modules, bounded on the right (i.e.,* $K^n = C^n = 0$ *for* $n \gg 0$*), inducing isomorphisms of cohomology (a quasiisomorphism, §19.2.3). Then "this quasiisomorphism commutes with arbitrary change of base ring": for every* B*-algebra* A*, the maps* $H^p(K^\bullet \otimes_B A) \to H^p(C^\bullet \otimes_B A)$ *are isomorphisms.*

Proof. The mapping cone $M^\bullet$ of $\alpha \colon K^\bullet \to C^\bullet$ is exact by Exercise 1.7.E. Then $M^\bullet \otimes_B A$ is still exact, by Exercise 25.3.F. But $M^\bullet \otimes_B A$ is the mapping cone of
$$\alpha \otimes_B A \colon K^\bullet \otimes_B A \to C^\bullet \otimes_B A,$$

so by Exercise 1.7.E, $\alpha \otimes_B A$ induces an isomorphism of cohomology (i.e., is a quasiisomorphism) too. □

Proof of Key Theorem 28.2.1. Choose a finite affine covering of X. Take the Čech complex $C^\bullet$ for $\mathscr{F}$ with respect to this cover. Recall that Grothendieck's Coherence Theorem 19.9.1 (which had Noetherian hypotheses) showed that the cohomology of $\mathscr{F}$ is coherent. (Theorem 19.9.1 required serious work. If you need Theorem 28.2.1 only in the projective case, the analogous statement with projective hypotheses, Theorem 19.8.1(d), was much easier.) Apply Lemma 28.2.2 to get the nicer variant $K^\bullet$ of the same complex $C^\bullet$. By Lemma 28.2.3, if we tensor with A and take cohomology, we get the same answer whether we use $K^\bullet$ or $C^\bullet$. □

We now use Theorem 28.2.1 to prove some of the fundamental results stated earlier: the Semicontinuity Theorem 28.1.1, Grauert's Theorem 28.1.5, and the Cohomology and Base Change Theorem 28.1.6. In the course of proving Semicontinuity, we will give a new proof of Theorem 25.7.1, that Euler characteristics are locally constant in flat families (that applies more generally in proper situations).

28.2.4. Proof of the Semicontinuity Theorem 28.1.1. The result is local on Y, so we may assume Y is affine. Let $K^\bullet$ be a complex as in Key Theorem 28.2.1.

Then for $q \in Y$,

$$\begin{aligned}\dim_{\kappa(q)} H^p(X_q, \mathscr{F}|_{X_q}) &= \dim_{\kappa(q)} \ker(\delta^p \otimes_B \kappa(q)) - \dim_{\kappa(q)} \operatorname{im}(\delta^{p-1} \otimes_B \kappa(q)) \\ &= \dim_{\kappa(q)}(K^p \otimes_B \kappa(q)) - \dim_{\kappa(q)} \operatorname{im}(\delta^p \otimes_B \kappa(q)) \\ &\qquad - \dim_{\kappa(q)} \operatorname{im}(\delta^{p-1} \otimes_B \kappa(q))\end{aligned} \tag{28.2.4.1}$$

Now $\dim_{\kappa(q)} \operatorname{im}(\delta^p \otimes_B \kappa(q))$ is a lower semicontinuous function on Y. (Reason: the locus where the dimension is less than some number N is obtained by setting all $N \times N$ minors of the matrix $K^p \to K^{p+1}$ to 0; cf. §12.4.4(ii)).) The same is true for $\dim_{\kappa(q)} \operatorname{im}(\delta^{p-1} \otimes_B \kappa(q))$. The result follows. □

28.2.5. A new proof (and extension to the proper case) of Theorem 25.7.1 that Euler characteristics of flat sheaves are locally constant.

If $K^\bullet$ were finite "on the left" as well — if $K^p = 0$ for $p \ll 0$ — then we would have a short proof of Theorem 25.7.1. By taking alternating sums (over p) of (28.2.4.1), we would have that

$$\chi(X_q, \mathscr{F}|_{X_q}) = \sum (-1)^p h^p(X_q, \mathscr{F}|_{X_q}) = \sum (-1)^p \operatorname{rank} K^p,$$

which is locally constant. The only problem is that the sums are infinite. We patch this problem by truncating the complex $K^\bullet$ below where there is cohomology. Define $J^\bullet$ by $J^p = K^p$ for $p \geq 0$, $J^p = 0$ for $p < -1$, and $J^{-1} := \ker(K^{-1} \to K^0)$. Then $J^\bullet$ is a complex in the obvious way, and the map of complexes $K^\bullet \to C^\bullet$ clearly factors through $J^\bullet$:

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & J^{-3} & \longrightarrow & J^{-2} & \longrightarrow & J^{-1} & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & J^2 & \longrightarrow & \cdots \\
 & & \| & & \| & & \| & & \| & & \| & & \| & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ker(K^{-1} \to K^0) & \longrightarrow & K^0 & \longrightarrow & K^1 & \longrightarrow & K^2 & \longrightarrow & \cdots \\
 & & \downarrow \sim & & \downarrow \sim & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \cdots
\end{array}$$

Clearly $J^\bullet \to C^\bullet$ induces an isomorphism on cohomology (recall both have 0 cohomology for $p < 0$).

Now J^{-1} (the kernel of a map of coherent modules) is coherent. Consider the mapping cone $M^\bullet$ of $\beta \colon J^\bullet \to C^\bullet$:

$$0 \to J^{-1} \to C^{-1} \oplus J^0 \to C^0 \oplus J^1 \to \cdots \to C^{n-1} \oplus J^n \to C^n \to 0.$$

From Exercise 1.7.E, as $J^\bullet \to C^\bullet$ induces an isomorphism on cohomology, the mapping cone has no cohomology — it is exact. All terms in it are flat except possibly J^{-1} (the C^p are flat by assumption, and J^i is free for $i \neq -1$). Hence J^{-1} is flat too, by Exercise 25.3.G. But flat coherent sheaves are locally free (Corollary 25.4.7). Then Theorem 25.7.1 follows from

$$\chi(X_q, \mathscr{F}|_{X_q}) = \sum (-1)^p h^p(X_q, \mathscr{F}|_{X_q}) = \sum (-1)^p \operatorname{rank} J^p.$$

$\square$

28.2.6. Proof of Grauert's Theorem 28.1.5 and the Cohomology and Base Change Theorem 28.1.6 (following Eric Larson).

($\star\star$ Experts: You might see in the proof that what makes pth cohomology commute with base change is that the complex of Key Theorem 28.2.1 can be "broken" into two complexes at the pth step. You might even want to interpret this in terms of the Čech complex as an object of the derived category of B-modules.)

Thanks to Theorem 28.2.1, Theorems 28.1.5 and 28.1.6 are now statements about a complex of free modules over a Noetherian ring.

28.2.7. *Definition.* Suppose $\phi \colon \mathscr{E} \to \mathscr{F}$ is a morphism of finite rank locally free sheaves on a scheme X. (More precisely: $\mathscr{E}$ and $\mathscr{E}$ are finite rank locally free sheaves, and $\phi \colon \mathscr{E} \to \mathscr{F}$ is a morphism of quasicoherent sheaves.) We say ϕ is **strongly of constant rank** a if for every point $p \in X$, there are integers b and c, and there is an open neighborhood U of p with a commutative diagram

$$\begin{array}{ccccc}
\mathscr{E}|_U & & \xrightarrow{\quad\phi\quad} & & \mathscr{F}|_U \\
\updownarrow \sim & & & & \updownarrow \sim \\
\mathscr{O}_U^{\oplus(a+b)} & \longrightarrow & \mathscr{O}_U^{\oplus a} & \longrightarrow & \mathscr{O}_U^{\oplus(a+c)}
\end{array}$$

where the map $\mathscr{O}_U^{\oplus(a+b)} \to \mathscr{O}_U^{\oplus a}$ is projection to the first a summands, and the map $\mathscr{O}_U^{\oplus a} \to \mathscr{O}_U^{\oplus(a+c)}$ is inclusion as the first a summands. We say that ϕ is **strongly of constant rank** if near every point $p \in X$, ϕ is strongly of constant rank a for some a.

28.2.8. *Important but straightforward observations.* (i) The notion "strongly of constant rank a" commutes with any base change $Y \to X$. (ii) The quasicoherent sheaves $\ker \phi$, $\operatorname{im} \phi$, and $\operatorname{coker} \phi$ are locally free (of finite rank b, a, and c respectively), and their construction commutes with any base change $Y \to X$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker \phi & \longrightarrow & \mathscr{E} & \xrightarrow{\phi} & \mathscr{F} & \longrightarrow & \operatorname{coker} \phi & \longrightarrow & 0 \\
 & & \updownarrow \wr & & \updownarrow \wr & & \updownarrow \wr & & \updownarrow \wr & & \\
0 & \longrightarrow & \mathscr{O}_U^{\oplus b} & \longrightarrow & \mathscr{O}_U^{\oplus (a+b)} & \longrightarrow & \mathscr{O}_U^{\oplus (a+c)} & \longrightarrow & \mathscr{O}_U^{\oplus c} & \longrightarrow & 0
\end{array}$$

(iii) If $\phi \colon \mathscr{E} \to \mathscr{F}$ is strongly of constant rank a, then at any point $p \in X$, the rank of $\phi|_p : \mathscr{E}_p \to |_p$ is a.

In the next exercises we give three criteria for when a morphism ϕ of finite rank locally free sheaves is strongly of constant rank.

28.2.A. EXERCISE. Suppose $\phi \colon \mathscr{E} \to \mathscr{F}$ is a morphism of finite rank locally free sheaves. Show that ϕ is strongly of constant rank a if and only if $\operatorname{coker} \phi$ is locally free of rank a.

28.2.B. EXERCISE. Suppose X is reduced, and $\phi \colon \mathscr{E} \to \mathscr{F}$ is a morphism of finite rank locally free sheaves. Show that ϕ is strongly of constant rank a if and only if $\phi|_p : \mathscr{E}|_p \to \mathscr{F}|_p$ is rank a for all points $p \in X$. (Hint: use the previous exercise, and Exercise 14.4.K.)

28.2.C. EXERCISE. Suppose $\phi \colon \mathscr{E} \to \mathscr{F}$ is a morphism of finite rank locally free sheaves, and $p \in X$. Show that the natural map $(\ker \phi)|_p \to \ker(\phi|_p)$ is surjective if and only if ϕ is strongly of constant rank in some neighborhood of p.

28.2.9. Proof of Grauert's Theorem 28.1.5. By hypothesis, $h^p(X_q, \mathscr{F}|_{X_q})$ is a locally constant function of $q \in Y$. From (28.2.4.1), $h^p(X_q, \mathscr{F}|_{X_q}) = \operatorname{rank} K^p - \operatorname{rank} \operatorname{im}(\delta^p|_q) - \operatorname{rank} \operatorname{im}(\delta^{p-1}|_q)$. But $\operatorname{rank} K^p$ is constant, and $\operatorname{rank} \operatorname{im}(\delta^p|_q)$ and $\operatorname{rank} \operatorname{im}(\delta^{p-1}|_q)$ are lower semicontinuous, so in fact $\operatorname{rank} \operatorname{im}(\delta^p|_q)$ and $\operatorname{rank} \operatorname{im}(\delta^{p-1}|_q)$ must be locally constant. By Exercise 28.2.B, both δ^{p-1} and δ^p are strongly of constant rank. Then by Observation 28.2.8(ii) $\operatorname{coker} \delta^{p-1}$ and $\operatorname{im} \delta^p$ are both locally free (of finite rank). In the short exact sequence

$$0 \to H^p(K^\bullet) \to \operatorname{coker} \delta^{p-1} \to \operatorname{im} \delta^p \to 0 \tag{28.2.9.1}$$

(Exercise 1.6.5.4, the "dual" definition of cohomology), both $\operatorname{coker} \delta^{p-1}$ and $\operatorname{im} \delta^p$ correspond to finite rank locally free sheaves. Thus $H^p(K^\bullet)$ does as well, by Exercise 14.0.F(a).

28.2.D. EXERCISE. Show (perhaps using (28.2.9.1)) that the construction of $H^p(K^\bullet)$ commutes with any base change, thereby completing the proof of Grauert's Theorem 28.1.5. □

In order to prove the Cohomology and Base Change Theorem 28.1.6, we need a preliminary result.

28.2.10. Lemma. — *Suppose*

$$\begin{array}{ccccc} K^{p-1} & \xrightarrow{\delta_K^{p-1}} & K^p & \xrightarrow{\delta_K^p} & K^{p+1} \\ \downarrow & & \downarrow & & \downarrow \\ J^{p-1} & \xrightarrow{\delta_J^{p-1}} & J^p & \xrightarrow{\delta_J^p} & J^{p+1} \end{array}$$

is a map of complexes, with the left vertical arrow surjective. Then $H^p(K^\bullet) \to H^p(J^\bullet)$ *is surjective if and only if* $\ker \delta_K^p \to \ker \delta_J^p$ *is surjective.*

Proof. The map $\operatorname{im} \delta_K^{p-1} \to \operatorname{im} \delta_J^{p-1}$ is surjective: any element α of $\operatorname{im} \delta_J^{p-1}$ lifts to J^{p-1}, then can lift to K^{p-1}, which then can map to K^p, which maps to α. Then apply the Snake Lemma 1.7.5 to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im} \delta_K^{p-1} & \longrightarrow & \ker \delta_K^p & \longrightarrow & H^p(K^\bullet) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{im} \delta_J^{p-1} & \longrightarrow & \ker \delta_J^p & \longrightarrow & H^p(J^\bullet) & \longrightarrow & 0. \end{array}$$

□

28.2.11. Proof of the Cohomology and Base Change Theorem 28.1.6. We focus on the complex near the pth step, and its restriction to the point $q \in X$:

$$\begin{array}{ccccc} K^{p-1} & \longrightarrow & K^p & \xrightarrow{\delta^p} & K^{p+1} \\ \downarrow & & \downarrow & & \downarrow \\ K^{p-1} \otimes_B \kappa(q) & \longrightarrow & K^p \otimes_B \kappa(q) & \xrightarrow{\delta^p \otimes \kappa(q)} & K^{p+1} \otimes_B \kappa(q) \end{array} \qquad \text{free } B\text{-modules}$$

By hypothesis, $\phi_q^p : (R^p \pi_* \mathscr{F})|_q \to H^p(X_q, \mathscr{F}|_{X_q})$ is surjective. By Lemma 28.2.10, this is equivalent to $(\ker \delta^p) \otimes \kappa(q) \to \ker(\delta^p \otimes \kappa(q))$ being surjective. By Exercise 28.2.C, this is equivalent to δ^p being strongly of constant rank near q, which implies (by Observation 28.2.8(ii)) that $\ker \delta^p$ is finite rank locally free, and the construction of $\ker \delta^p$ commutes with any base change in a neighborhood of q.

Now $H^p(K^\bullet) = \operatorname{coker}(K^{p-1} \to \ker \delta^p)$, i.e., we have

$$K^{p-1} \longrightarrow \ker \delta^p \longrightarrow H^p(K^\bullet) \longrightarrow 0.$$

Thus $H^p(K^\bullet)$ commutes with any base change (as tensor product is right exact). This completes the proof of part (i) of the Theorem.

For part (ii), consider again the map $K^{p-1} \to \ker \delta^p$ of finite rank locally free sheaves, whose cokernel is $H^p(K^\bullet)$. Now $H^p(K^\bullet)$ is locally free if and only if $K^{p-1} \to \ker \delta^p$ is strongly of constant rank, if and only if (since δ^p is strongly of constant rank) δ^{p-1} is strongly of constant rank, if and only if $H^{p-1}(K^\bullet) \to H^{p-1}(K^\bullet|_q)$ is surjective. This completes the proof of part (ii). □

28.2.12. ⋆ **Removing Noetherian conditions.**

It can be helpful to have versions of the theorems of §28.1 without Noetherian conditions; important examples come from moduli theory, and will be discussed in the next section. Noetherian conditions can often be exchanged for finite presentation conditions. We begin with an extension of Exercise 10.3.H.

28.2.E. EXERCISE. Suppose $\pi \colon X \to \operatorname{Spec} B$ is a finitely presented morphism, and $\mathscr{F}$ is a finitely presented sheaf on X. Show that there exists a base change diagram of the form

(28.2.12.1)
$$\begin{array}{ccc} \mathscr{F} & & \mathscr{F}' \\ | & & | \\ X & \xrightarrow{\sigma} & X' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \operatorname{Spec} B & \xrightarrow{\rho} & \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_N]/I \end{array}$$

where N is some integer, $I \subset \mathbb{Z}[x_1, \dots, x_N]$, and π' is finitely presented (= finite type as the target is Noetherian, see §8.3.13), and a finitely presented (= coherent) quasicoherent sheaf $\mathscr{F}'$ on X' with $\mathscr{F} \cong \sigma^* \mathscr{F}'$.

28.2.13. *Properties of* π'. (The ideal I appears in the statement of Exercise 28.2.E not because it is needed there, but to make the statement of this remark correct.) If π is proper, then diagram (28.2.12.1) can be constructed so that π' is also proper (using [**Gr-EGA**, IV_3.8.10.5]). Furthermore, if $\mathscr{F}$ is flat over $\operatorname{Spec} B$, then (28.2.12.1) can be constructed so that $\mathscr{F}'$ is flat over $\operatorname{Spec} \mathbb{Z}[x_1, \dots, x_N]/I$ (using [**Gr-EGA**, IV_3.11.2.6]). This requires significantly more work.

28.2.F. EXERCISE. *Assuming* the results stated in §28.2.13, prove the following results, with the "locally Noetherian" hypotheses removed, and "finite presentation" hypotheses added:

(a) the constancy of Euler characteristic in flat families (Theorem 25.7.1, extended to the proper case as in §28.2.5);
(b) the Semicontinuity Theorem 28.1.1;
(c) Grauert's Theorem 28.1.5 (you will have to show that $\mathbb{Z}[x_1, \dots, x_N]/I$ in (28.2.12.1) can be taken to be reduced); and
(d) the Cohomology and Base Change Theorem 28.1.6.

28.2.14. *Necessity of finite presentation conditions.* The finite presentation conditions are necessary. There is a projective flat morphism to a connected target where the fiber dimension jumps. There is a finite flat morphism where the degree of the fiber is not locally constant. There is a projective flat morphism to a connected target where the fibers are curves, and the arithmetic genus is not constant. See [**Stacks**, tag 05LB] for the first example; the other two use the same idea.

28.3 Applying cohomology and base change to moduli problems

The theory of moduli relies on ideas of cohomology and base change. We explore this by examining two special cases of one of the primordial moduli spaces, the Hilbert scheme: the Grassmannian, and the fact that degree d hypersurfaces in projective space are "parametrized" by another projective space (corresponding to degree d polynomials, see Remark 4.5.3).

As suggested in §25.1, the Hilbert functor $\mathrm{Hilb}_Y \mathbb{P}^n$ of $\mathbb{P}^n_Y$ parametrizes finitely presented closed subschemes of $\mathbb{P}^n_Y$, where Y is an arbitrary scheme. More precisely, it is a contravariant functor sending the Y-scheme X to the set of finitely presented closed subschemes of $X \times_Y \mathbb{P}^n_Y = \mathbb{P}^n_X$ flat over X (and sending morphisms $X_1 \to X_2$ to pullbacks of flat families). An early achievement of Grothendieck was the construction of the Hilbert scheme, which can then be cleverly used to construct many other moduli spaces.

28.3.1. Theorem (Grothendieck). — $\mathrm{Hilb}_{\mathbb{Z}} \mathbb{P}^n$ *is representable by a scheme locally of finite type.*

(Grothendieck's original argument is in [**Gr5**]. A readable construction is given in [**Mu2**], and in [**FGIKNV**, Ch. 5].)

28.3.A. EASY EXERCISE. Assuming Theorem 28.3.1, show that $\mathrm{Hilb}_Y \mathbb{P}^n$ is representable, by showing that it is represented by $\mathrm{Hilb}_{\mathbb{Z}} \mathbb{P}^n \times_{\mathbb{Z}} Y$. Thus the general case follows from the "universal" case of $Y = \mathbb{Z}$.

28.3.B. EXERCISE. Assuming Theorem 28.3.1, show that $\mathrm{Hilb}_{\mathbb{Z}} \mathbb{P}^n$ is the disjoint union of schemes $\mathrm{Hilb}^{p(m)}_{\mathbb{Z}} \mathbb{P}^n$, each one corresponding to finitely presented closed subschemes of $\mathbb{P}^n_{\mathbb{Z}}$ whose fibers have fixed Hilbert polynomial $p(m)$. Hint: Corollary 25.7.2.

28.3.2. Theorem (Grothendieck). — *Each* $\mathrm{Hilb}^{p(m)}_{\mathbb{Z}} \mathbb{P}^n$ *is projective over* $\mathbb{Z}$.

In order to get some feeling for the Hilbert scheme, we discuss two important examples, without relying on Theorem 28.3.1.

28.3.3. The Grassmannian.

We have defined the Grassmannian $G(k, n)$ twice before, in §7.7 and §17.7. The second time involved showing the representability of a (contravariant) functor (from *Sheaves* to *Sets*), of rank k locally free quotient sheaves of a rank n free sheaf.

We now consider a parameter space for a more geometric problem. The space will again be $G(k, n)$, but because we won't immediately know this, we invent some temporary notation. Let $G'(k, n)$ be the contravariant functor (from *Schemes* to *Sets*) which assigns to a scheme B the set of *finitely presented* closed subschemes of $\mathbb{P}^{n-1}_B$, flat over B, whose fiber over any point $b \in B$ is a (linearly embedded) $\mathbb{P}^{k-1}_{\kappa(b)}$ in $\mathbb{P}^{n-1}_{\kappa(b)}$:

(28.3.3.1)
$$\begin{array}{ccc} X & \xrightarrow{\text{cl. subscheme}} & \mathbb{P}^{n-1}_B \\ {\scriptstyle \text{flat, f. pr.}}\downarrow & \swarrow{\scriptstyle \pi} & \\ B & & \end{array}$$

(This describes the map to *Sets*; you should think through how pullback makes this into a contravariant functor.)

28.3.4. Theorem. — *The functor* $G'(k, n)$ *is represented by* $G(k, n)$.

Translation: there is a natural bijection between diagrams of the form (28.3.3.1) (where the fibers are $\mathbb{P}^{k-1}$'s) and diagrams of the form (17.7.0.1) (the diagrams that $G(k,n)$ parametrizes, or represents).

One direction is notably easier. Suppose we are given a diagram of the form (17.7.0.1) over a scheme B,

$$\mathscr{O}_B^{\oplus n} \twoheadrightarrow \mathscr{Q}, \tag{28.3.4.1}$$

where $\mathscr{Q}$ is locally free of rank k. Applying $\mathcal{P}roj_B$ to the $\mathrm{Sym}^\bullet$ construction on both $\mathscr{O}_B^{\oplus n}$ and $\mathscr{Q}$, we obtain a closed embedding

$$\begin{array}{ccccc} \mathcal{P}roj_B\,(\mathrm{Sym}^\bullet \mathscr{Q}) & \hookrightarrow & \mathcal{P}roj_B\left(\mathrm{Sym}^\bullet \mathscr{O}_B^{\oplus n}\right) & = & \mathbb{P}^{n-1} \times B \\ & \searrow & \swarrow & & \\ & B & & & \end{array} \tag{28.3.4.2}$$

(as, for example, in Exercise 18.2.H).

The fibers are linearly embedded $\mathbb{P}^{k-1}$'s (as base change, in this case to a point of B, commutes with the $\mathcal{P}roj$ construction, Exercise 18.2.E). Note that $\mathcal{P}roj(\mathrm{Sym}^\bullet \mathscr{Q})$ is flat and finitely presented over B, as it is a projective bundle. We have constructed a diagram of the form (28.3.3.1).

We now need to reverse this. The trick is to produce (28.3.4.1) from our geometric situation (28.3.3.1), and this is where cohomology and base change will be used.

Given a diagram of the form (28.3.3.1) (where the fibers are $\mathbb{P}^{k-1}$'s), consider the closed subscheme exact sequence for X:

$$0 \longrightarrow \mathscr{I}_X \longrightarrow \mathscr{O}_{\mathbb{P}_B^{n-1}} \longrightarrow \mathscr{O}_X \longrightarrow 0.$$

Tensor this with $\mathscr{O}_{\mathbb{P}_B^{n-1}}(1)$:

$$0 \longrightarrow \mathscr{I}_X(1) \longrightarrow \mathscr{O}_{\mathbb{P}_B^{n-1}}(1) \longrightarrow \mathscr{O}_X(1) \longrightarrow 0. \tag{28.3.4.3}$$

Note that $\mathscr{O}_X(1)$ restricted to each fiber of π is $\mathscr{O}(1)$ on $\mathbb{P}^{k-1}$ (over the residue field), for which all higher cohomology vanishes (§19.3).

28.3.C. EXERCISE. Show that $R^i\pi_*\mathscr{O}_X(1) = 0$ for $i > 0$, and $\pi_*\mathscr{O}_X(1)$ is locally free of rank k. Hint: use the Cohomology and Base Change Theorem 28.1.6. Either use the non-Noetherian discussion of §28.2.12 (which we haven't proved), or else just assume B is locally Noetherian.

28.3.D. EXERCISE. Show that the long exact sequence obtained by applying π_* to (28.3.4.3) is just a short exact sequence of locally free sheaves

$$0 \longrightarrow \pi_*\mathscr{I}_X(1) \longrightarrow \pi_*\mathscr{O}_{\mathbb{P}_B^{n-1}}(1) \longrightarrow \pi_*\mathscr{O}_X(1) \longrightarrow 0.$$

of ranks $n-k$, n, and k respectively, where the middle term is canonically identified with $\mathscr{O}_B^{\oplus n}$.

The surjection $\mathscr{O}_B^{\oplus n} \twoheadrightarrow \pi_*\mathscr{O}_X(1)$ is precisely a diagram of the sort we wished to construct, (17.7.0.1).

28.3.E. EXERCISE. Close the loop, by using these two "inverse" constructions to show that $G(k,n)$ represents the functor $G'(k,n)$.

28.3.5. Hypersurfaces.

Ages ago (in Remark 4.5.3), we informally said that hypersurfaces of degree d in $\mathbb{P}^n$ are parametrized by a $\mathbb{P}^{\binom{n+d}{d}-1}$. We now make this precise. We work over a base $\mathbb{Z}$ for suitable generality. You are welcome to replace $\mathbb{Z}$ by a field of your choice, but by the same argument as in Easy Exercise 28.3.A, all other cases are obtained from this one by base change.

Define the contravariant functor $H_{d,n} : \mathit{Sch} \to \mathit{Sets}$ from schemes to sets as follows. To a scheme B, we associated the set of all closed subschemes $X \hookrightarrow \mathbb{P}^n_B$, flat and finitely presented over B, all of whose fibers are degree d hypersurfaces in $\mathbb{P}^n$ (over the appropriate residue field). To a morphism $B_1 \to B_2$, we obtain a map $H_{d,n}(B_2) \to H_{d,n}(B_1)$ by pullback.

28.3.6. Proposition. — *The functor $H_{d,n}$ is represented by $\mathbb{P}^{\binom{n+d}{d}-1}$.*

As with the case of the Grassmannian, one direction is easy, and the other requires cohomology and base change.

28.3.F. EASY EXERCISE. Over $\mathbb{P}^{\binom{n+d}{d}-1}$, described a closed subscheme $\mathscr{X} \hookrightarrow \mathbb{P}^n \times \mathbb{P}^{\binom{n+d}{d}-1}$ that will be the universal hypersurface. Show that $\mathscr{X}$ is flat and finitely presented over $\mathbb{P}^{\binom{n+d}{d}-1}$. (For flatness, you can use the local criterion of flatness on the source, Exercise 25.6.F, but it is possible to deal with it easily by working by hand.)

Thus given any morphism $B \to \mathbb{P}^{\binom{n+d}{d}-1}$, by pullback, we have a degree d hypersurface X over B (an element of $H_{d,n}(B)$).

Our goal is to reverse this process: from a degree d hypersurface $\pi\colon X \to \mathbb{P}^n_B$ over B (an element of $H_{d,n}(B)$), we want to describe a morphism $B \to \mathbb{P}^{\binom{n+d}{d}-1}$.

Consider the closed subscheme exact sequence for $X \hookrightarrow \mathbb{P}^n_B$, twisted by $\mathscr{O}_{\mathbb{P}^n_B}(d)$:

$$0 \longrightarrow \mathscr{I}_X(d) \longrightarrow \mathscr{O}_{\mathbb{P}^n_B}(d) \longrightarrow \mathscr{O}_X(d) \longrightarrow 0. \tag{28.3.6.1}$$

28.3.G. EXERCISE (CF. EXERCISE 28.3.C). Show that the higher pushforwards (by π) of each term of (28.3.6.1) is 0, and that the long exact sequence of pushforwards of (28.3.6.1) is

$$0 \longrightarrow \pi_*\mathscr{I}_X(d) \longrightarrow \pi_*\mathscr{O}_{\mathbb{P}^n_B}(d) \longrightarrow \pi_*\mathscr{O}_X(d) \longrightarrow 0.$$

where the middle term is free of rank $\binom{n+d}{d}$ (whose summands can be identified with degree d monomials in the projective variables $x_1, \dots, x_n$ (see Exercise 9.3.J), and the left term $\pi_*\mathscr{I}_X(d)$ is locally free of rank 1 (basically, a line bundle).

(It is helpful to interpret the middle term $\mathscr{O}_B^{\oplus\binom{n+d}{d}}$ as parametrizing homogeneous degree d polynomials in $n+1$ variables, and the rank 1 subsheaf of $\pi_*\mathscr{I}_X(d)$ as "the equation of X". This will motivate what comes next.)

Taking the dual of the injection $\pi_* \mathscr{I}_X(d) \hookrightarrow \mathscr{O}_B^{\oplus \binom{n+d}{d}}$, we have a surjection

$$\mathscr{O}_B^{\oplus \binom{n+d}{d}} \twoheadrightarrow \mathscr{L}$$

from a free sheaf onto an invertible sheaf $\mathscr{L} = (\pi_* \mathscr{I}_X(d))^\vee$, which (by the universal property of projective space) yields a morphism $B \to \mathbb{P}^{\binom{n+d}{d}-1}$.

28.3.H. EXERCISE. Close the loop: show that these two constructions are inverses, thereby proving Proposition 28.3.6.

28.3.7. *Remark.* The proof of the representability of the Hilbert scheme shares a number of features of our arguments about the Grassmannian and the parameter space of hypersurfaces.