from Eric Larson

(scribbles by Ravi Vakil)

Cohomology and Base Change

This is intended to be an alternative to what is done in 28.2.6 of your book ("Proof of Grauert's Theorem 28.1.5 and the Cohomology and Base Change Theorem 28.1.6").

Maps between vector bundles of constant rank

Let $f\colon \mathcal{E} \to \mathcal{F}$ be a map between locally free sheaves on a scheme X.

Definition 1. We say f is *weakly of constant rank* a if for any point $x \in X$, the restriction to the fiber $f|_x\colon \mathcal{E}|_x \to \mathcal{F}|_x$ is of rank a.

We say f is *strongly of constant rank* a if for any $x \in X$, there is an open neighborhood U of x such that $f|_U\colon \mathcal{E}|_U \to \mathcal{F}|_U$ is isomorphic to the map $\mathcal{O}_U^{a+b} \to \mathcal{O}_U^{a+c}$, given by the composition of projection and inclusion $\mathcal{O}_U^{a+b} \to \mathcal{O}_U^{a} \to \mathcal{O}_U^{a+c}$.

If f is strongly of constant rank, then f is weakly of constant rank; moreover, the image, kernel, and cokernel of f are locally free, and their formation commutes with basechange. The property of being strongly of constant rank can be detected using either the kernel or cokernel, as follows:

residue field at x

Lemma 2. *The cokernel of f is locally free if and only if f is strongly of constant rank.*

Proof. The "if" is clear. For the "only if", pick any $x \in X$, and choose bases $\mathcal{E}|_x \simeq \underbrace{k(x)}_{}{}^{a+b}$ and $\mathcal{F}|_x \simeq k(x)^{a+c}$ under which the map $f|_x\colon \mathcal{E}|_x \to \mathcal{F}|_x$ is given by the composition of projection and inclusion.

Lift the first a basis elements of $\mathcal{E}|_x$ and last c basis elements of $\mathcal{F}|_x$ arbitrarily to sections $e_1, \ldots, e_a$ of $\mathcal{E}$ and $f_1, \ldots, f_c$ of $\mathcal{F}$ respectively. Since $\mathcal{F}$ is locally free, $f(e_1), \ldots, f(e_a), f_1, \ldots, f_c$ freely generate $\mathcal{F}$ in a neighborhood of x. Since $\operatorname{coker} f$ is locally free, $f_1, \ldots, f_c$ are linearly independent in $\operatorname{coker} f$ near x, and thus $f(e_1), \ldots, f(e_a)$ generate $\operatorname{im} f$ near x.

Then lift the second b basis elements of $\mathcal{E}|_x$ to sections $k_1, \ldots, k_b$ of $\mathcal{E}$. Since $f(e_1), \ldots, f(e_a)$ generate $\operatorname{im} f$ near x, we may write $f(k_i) = \sum a_{ij} f(e_j)$, and correct the k_i to $k_i' = k_i - \sum a_{ij} e_j$. Since $\mathcal{E}$ is locally free, $k_1', \ldots, k_b', e_1, \ldots, e_a$ freely generate $\mathcal{E}$ in a neighborhood of x.

The sections e_i, k_i' and $f(e_i), f_i$ then give isomorphisms $\mathcal{E} \simeq \mathcal{O}^{a+b}$ and $\mathcal{F} \simeq \mathcal{O}^{a+c}$ under which f takes the desired form. □

Corollary 3. *If X is reduced, the notions of weakly and strongly of constant rank coincide.*

Proof. Suppose that f is weakly of constant rank. Since restriction is right exact, formation of coker commutes with restriction. This implies $\operatorname{coker} f$ is a coherent sheaf of constant rank, thus locally free. Applying the lemma, we have that f is strongly of constant rank. □

Lemma 4. *Fix a point $x \in X$. Then $\ker(f) \to \ker(f|_x)$ is surjective if and only if f is strongly of constant rank in a neighborhood of x.*

Proof. The "if" is clear. For the "only if", choose bases $\mathcal{E}|_x \simeq k(x)^{a+b}$ and $\mathcal{F}|_x \simeq k(x)^{a+c}$ under which the map $f|_x\colon \mathcal{E}|_x \to \mathcal{F}|_x$ is given by the composition of projection and inclusion.

Lift the first a basis elements of $\mathcal{E}|_x$ and last c basis elements of $\mathcal{F}|_x$ arbitrarily to sections $e_1, \ldots, e_a$ of $\mathcal{E}$ and $f_1, \ldots, f_c$ of $\mathcal{F}$ respectively. Since $\ker(f) \to \ker(f|_x)$ is surjective, we may lift the second b basis elements of $\mathcal{E}|_x$ to sections $k_1, \ldots, k_b \in \ker(f)$.

The sections e_i, k_i and $f(e_i), f_i$ then give isomorphisms $\mathcal{E} \simeq \mathcal{O}^{a+b}$ and $\mathcal{F} \simeq \mathcal{O}^{a+c}$ in a neighborhood of x, under which f takes the desired form. □

Proof of Grauert's Theorem

By the proof of the semicontinuity theorem, $K^{p-1} \to K^p$ and $K^p \to K^{p+1}$ are weakly of constant rank. By Corollary 3, they are strongly of constant rank, which implies Grauert's Theorem.

Proof of Cohomology and Base Change

Lemma 5. *If $K^\bullet \to J^\bullet$ is a surjective map of complexes (termwise), then $H^p(K^\bullet) \to H^p(J^\bullet)$ is surjective if and only if $\ker \delta_K^p \to \ker \delta_J^p$ is surjective.*

Proof. Since $K^\bullet \to J^\bullet$ is surjective, $\operatorname{im} \delta_K^{p-1} \to \operatorname{im} \delta_J^{p-1}$ is surjective. The result now follows from the ~~five~~ the snake lemma applied to:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im} \delta_K^{p-1} & \longrightarrow & \ker \delta_K^p & \longrightarrow & H^p(K^\bullet) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{im} \delta_J^{p-1} & \longrightarrow & \ker \delta_J^p & \longrightarrow & H^p(J^\bullet) & \longrightarrow & 0. \end{array}$$

□

Thus ~~that~~ if $K^\bullet$ is a complex of vector bundles on X, the restriction map $H^p(K^\bullet) \to H^p(K|_x^\bullet)$ is surjective if and only if $\delta^p \colon K^p \to K^{p+1}$ is strongly of constant rank in a neighborhood of x by Lemmas 5 and 4. If this is the case, then:

1. In particular, formation of $\ker \delta^p$ commutes with basechange. Since formation of coker always commutes with basechange, formation of $H^p(K^\bullet) = \operatorname{coker}(K^{p-1} \to \ker \delta^p)$ commutes with basechange.

2. Moreover, by Lemma 2, $H^p(K^\bullet)$ is locally free in a neighborhood of x if and only if $K^{p-1} \to \ker \delta^p$ is strongly of constant rank in a neighborhood of x, or equivalently (since δ^p is strongly of constant rank), if and only if δ^{p-1} is strongly of constant rank in a neighborhood of x. But we already showed that this is equivalent to the surjectivity of $H^{p-1}(K^\bullet) \to H^{p-1}(K|_x^\bullet)$.

Bonus material (part of Eric's motivation)

Jumping Behavior on a Curve

If Y is a smooth curve, then since $\ker \delta^p$ and $\operatorname{im} \delta^p$ are subsheaves of a locally free sheaf, they are locally free. In particular, $\ker \delta^p \to K^p$ is strongly of constant rank by Lemma 2 (as its cokernel is $\operatorname{im} \delta^p$). The rank of $\ker \delta^p$ is thus constant, and the rank of $K^{p-1} \to \ker \delta^p$ coincides with the rank of δ^{p-1}. Since formation of cokernels commute with restriction, we have

$$\operatorname{rk} H^p(K^\bullet)|_x = \operatorname{rk} K^p - \operatorname{rk} \delta^p - \operatorname{rk}(\delta^{p-1}|_x).$$

This should be compared with the formula obtained in the proof of the semicontinuity theorem:

$$\operatorname{rk} H^p(K^\bullet|_x) = \operatorname{rk} K^p - \operatorname{rk}(\delta^p|_x) - \operatorname{rk}(\delta^{p-1}|_x).$$