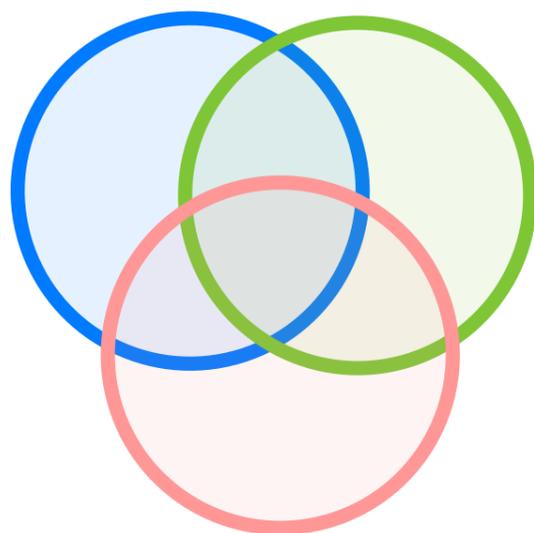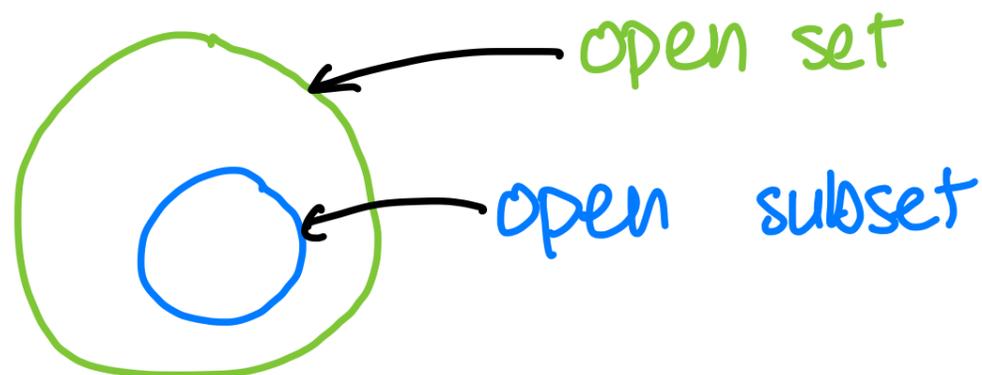# Puzzling through exact sequences

A Bedtime Story with Pictures

Ravi Vakil

August 9, 2021.

When I was younger, I learned that Venn diagrams were a helpful way of visualizing sets.
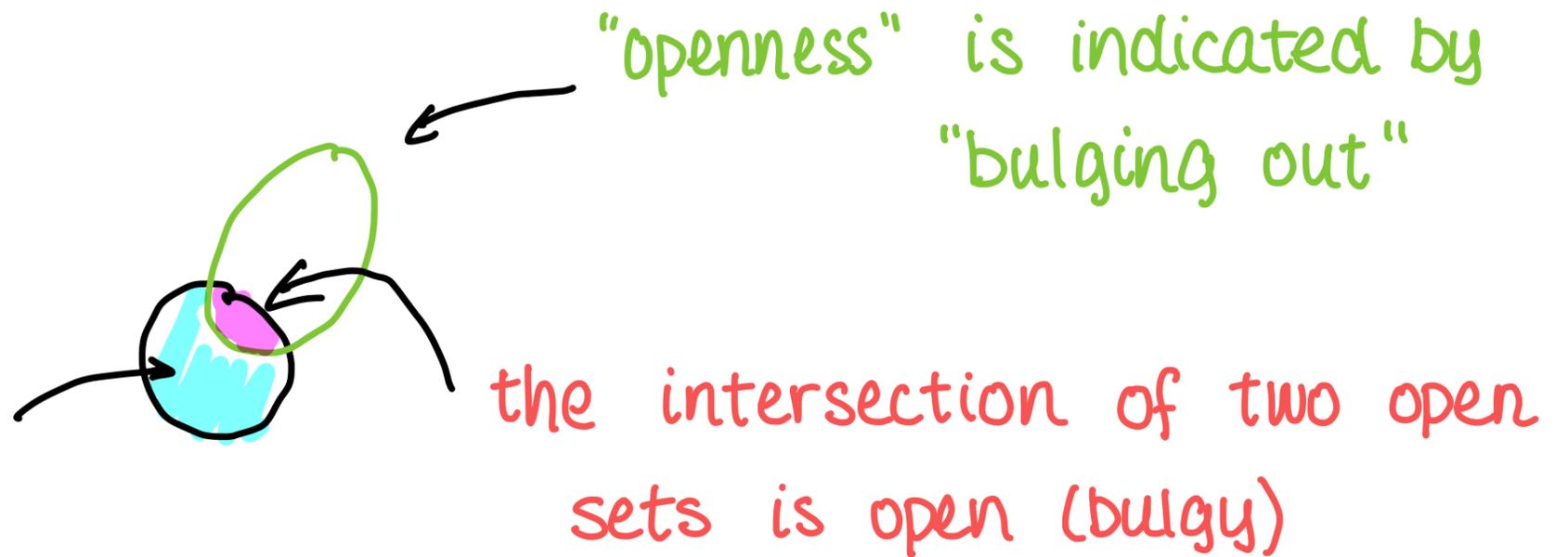
Later, when learning point-set topology, I extend this to picture open subsets of a topological space.
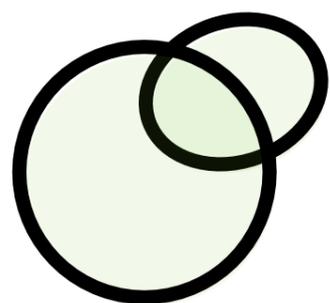
open set

open subset

It helped me understand and "see" properties of open sets.

"openness" is indicated by "bulging out"

the intersection of two open sets is open (bulgy)
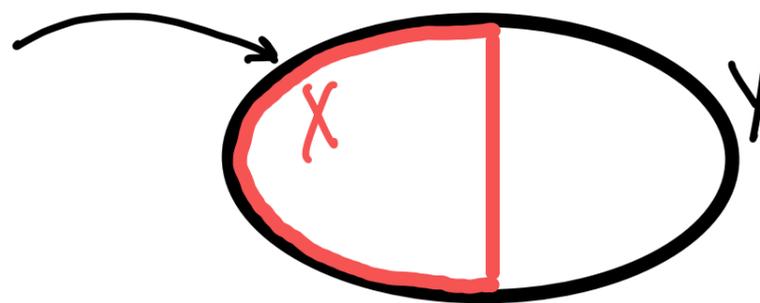
this is not open (not bulgy)

This required refining my notion of "bulginess" to make it work.
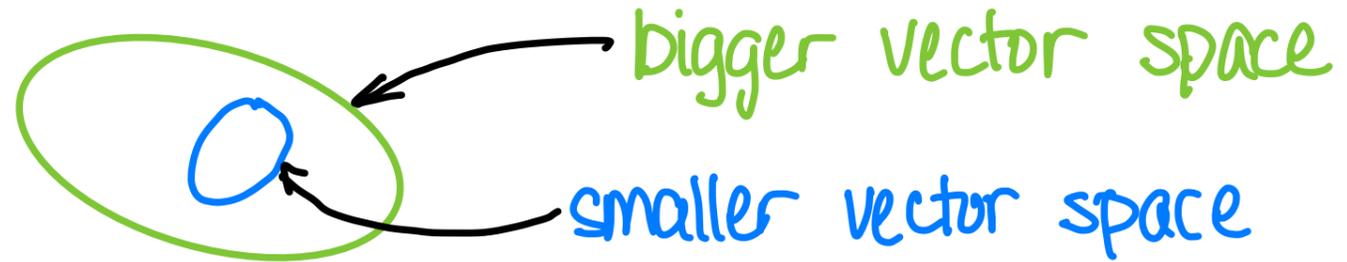
← the union of two open sets is open (bulgy)

X is both closed and open in Y.

X Y

But once it worked, it worked pretty well.

Something similar works in linear algebra.

a first
approximation:

bigger vector space
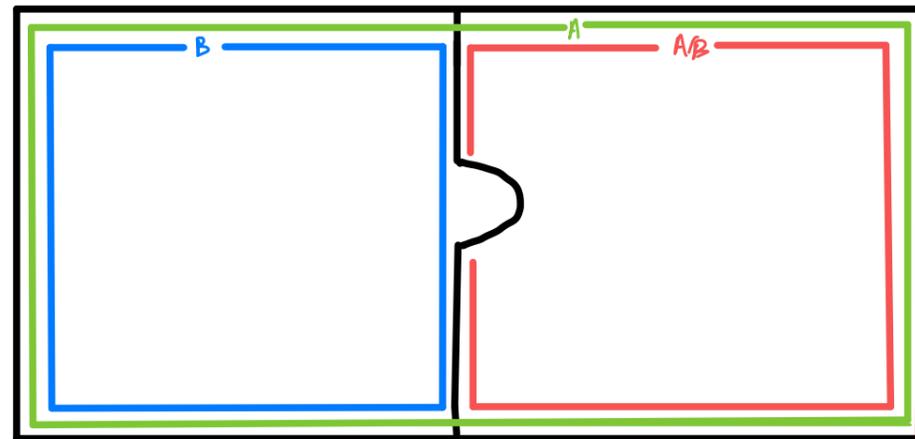
smaller vector space

So how do we hardwire quotients into the picture?

One possibility: by interpreting the quotient A/B as the complement of

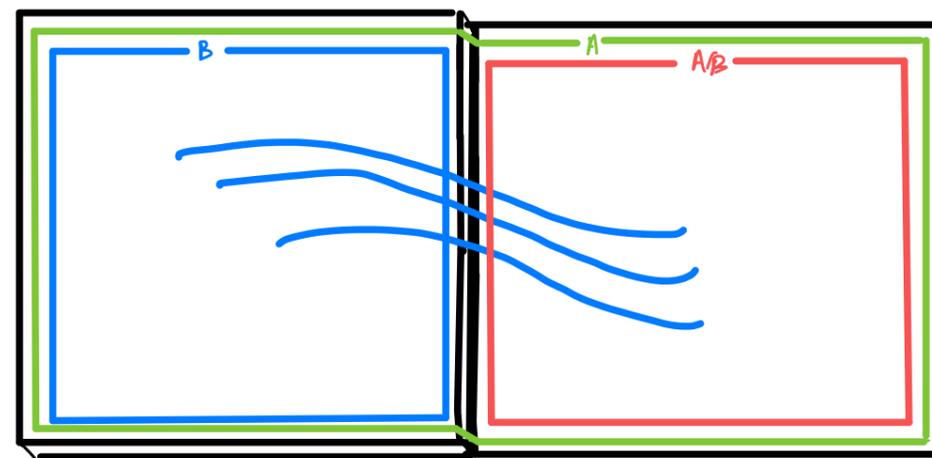bulgy shape B in bulgy shape A.

A

B

A/B
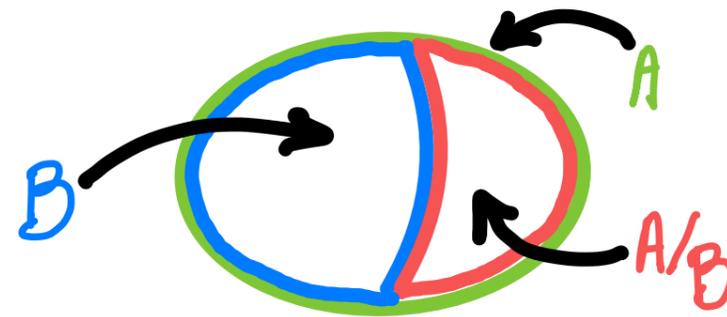
We visually encode that B is a subspace of A, and A/B is a quotient of A.

Once things get sufficiently complicated, it becomes hard to draw bulges. Instead, we use the metaphor of jigsaw puzzles

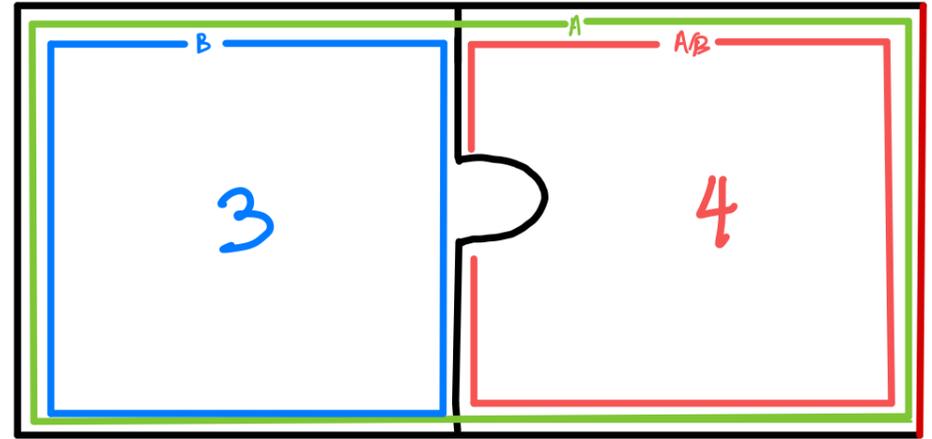I also like to picture the "subs" as at a higher elevation than the "quotients", with water gently flowing downhill from "sub" areas to "quotient" areas.

We can also decorate the regions, for example by dimension.

Dimension is additive by region

so here, dim $A = 7$

Here we see that if V_1 and V_2 are subspaces of a vector space W, then

$$\dim (V_1 + V_2) = \dim V_1 + \dim V_2 - \dim V_1 \cap V_2$$
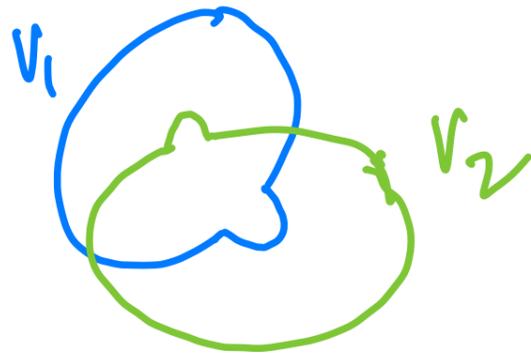
> This works more generally ~~for abelian groups~~
> ~~for modules over a given ring~~
> for objects in an abelian category.

In order to not distract you with fancy words, I will use the language of abelian groups, but you may secretly pretend that we are talking in as much generality as you feel comfortable with.

Here is an abelian group A, with a subgroup B, and the corresponding quotient group C.

B →

A

$C = A/B$
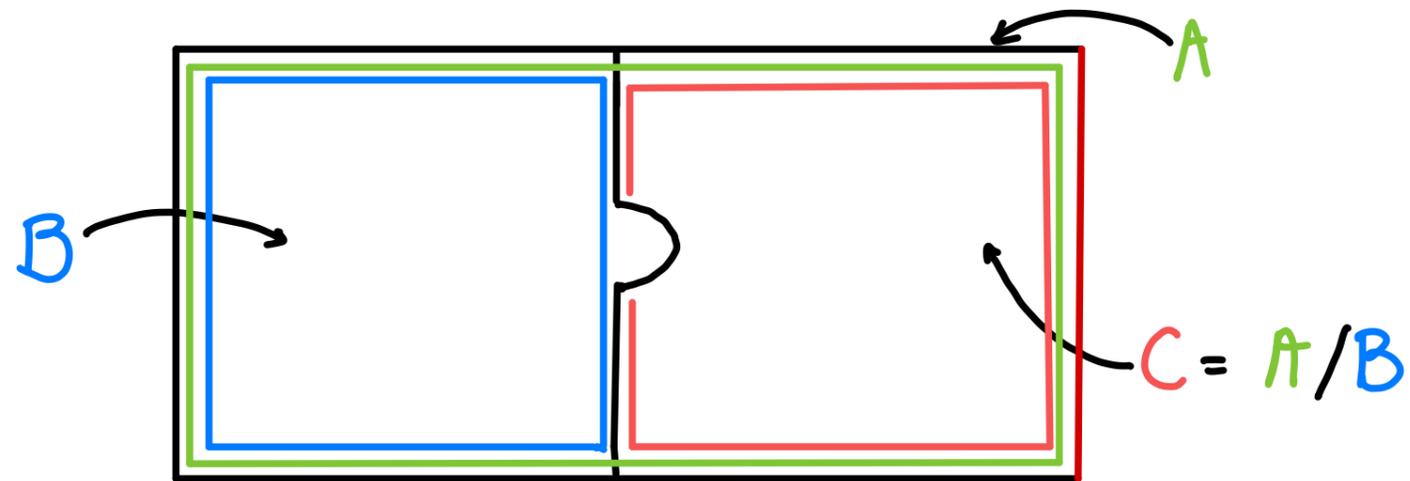
If $A = B \oplus C$, we might instead draw

B →

$A = B \oplus C$

C
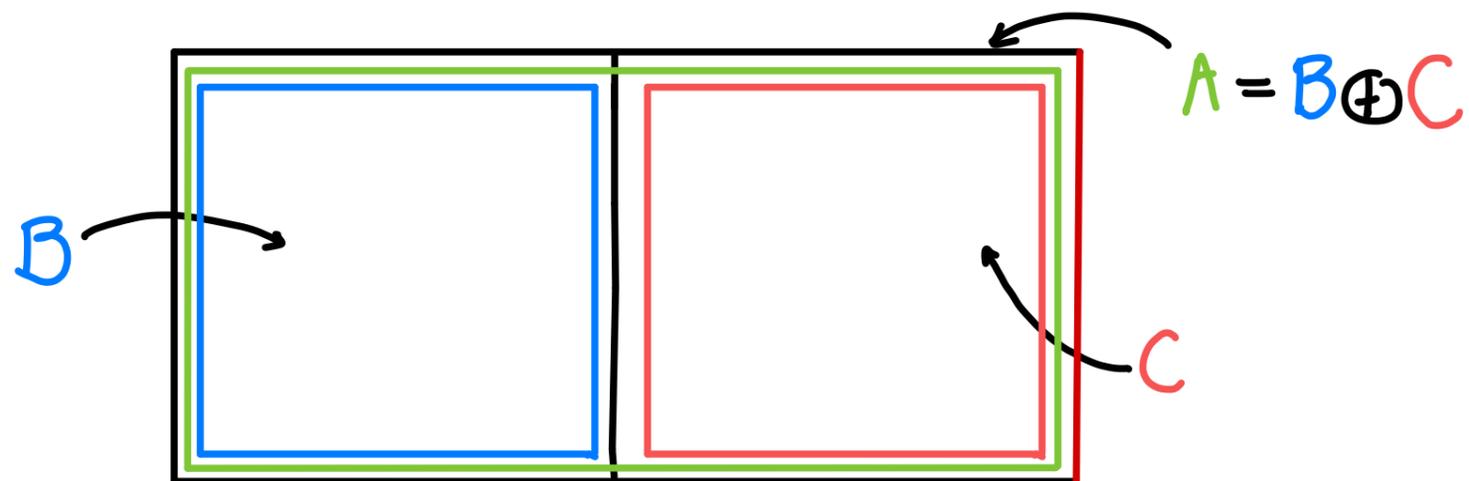
Here is a length three filtration of A: $0 \subset C \subset B \subset A$.
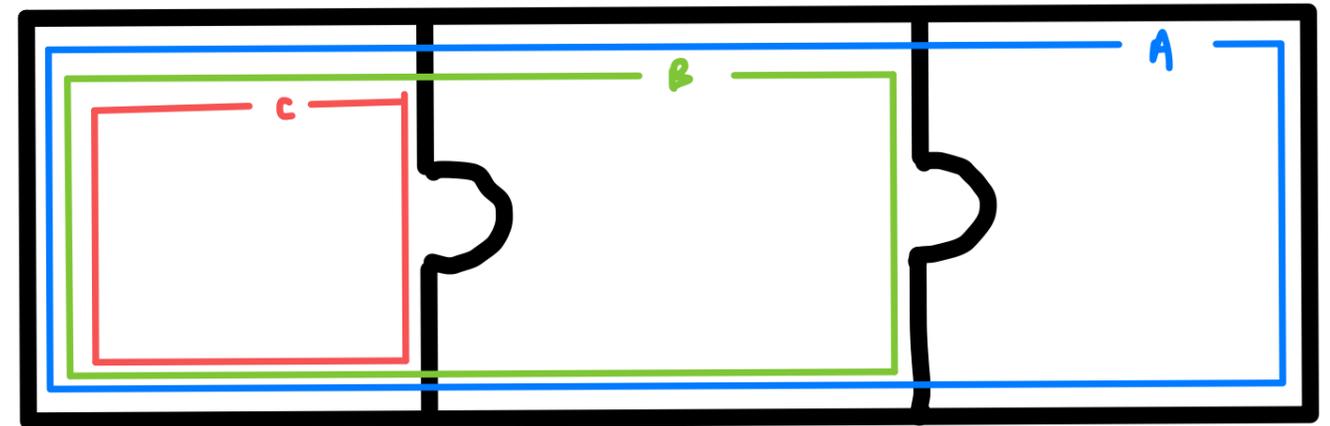
It is also, basically immediately, a picture of the

"Third Isomorphism Theorem":

If $C \subset B \subset A$, then $B/C \subset A/C$, and A/B is canonically identified with $\frac{A/C}{B/C}$.
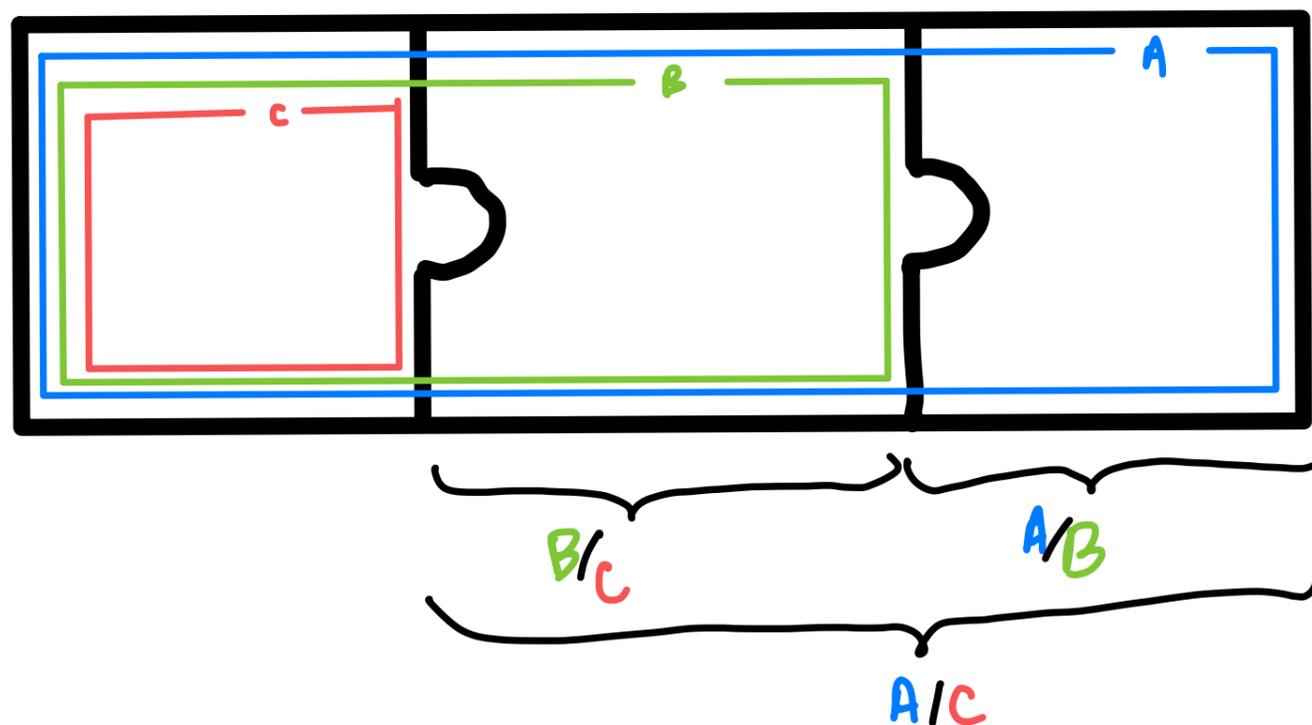
To summarize, we have a picture:
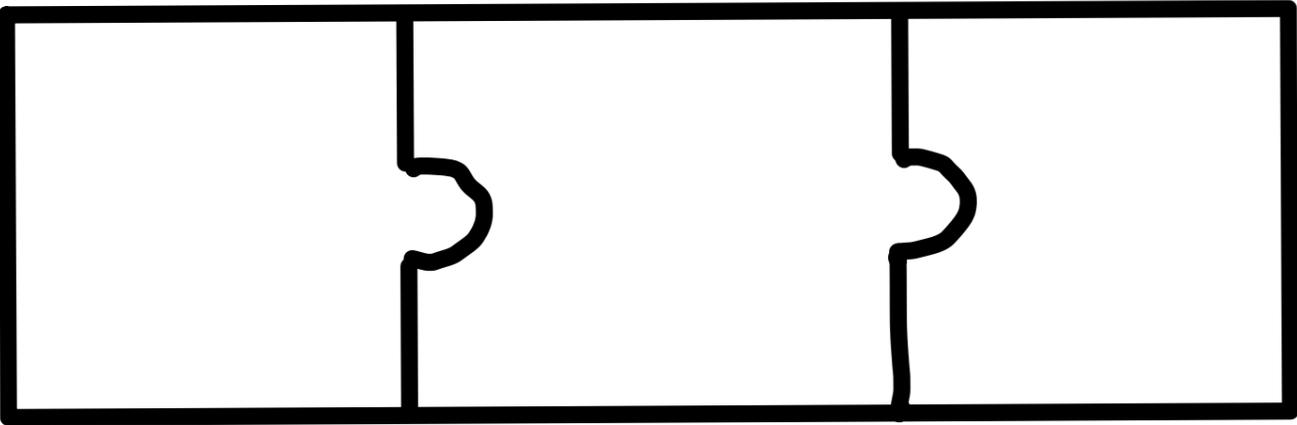
along with some labels

which tell us which combinations of pieces "we can discuss" (or if you prefer, to which we ascribe meaning):
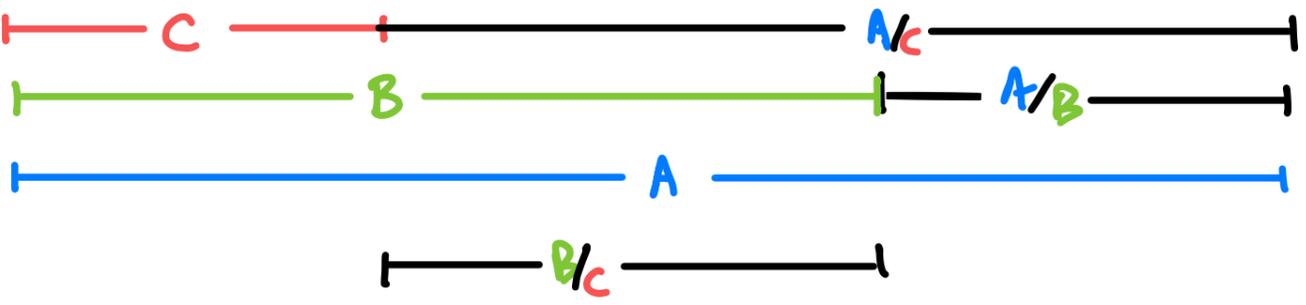
which in turn encodes relationships among these groups.

Here is a $\boxed{\text{group homomorphism}}$ $\phi: X \longrightarrow Y$.
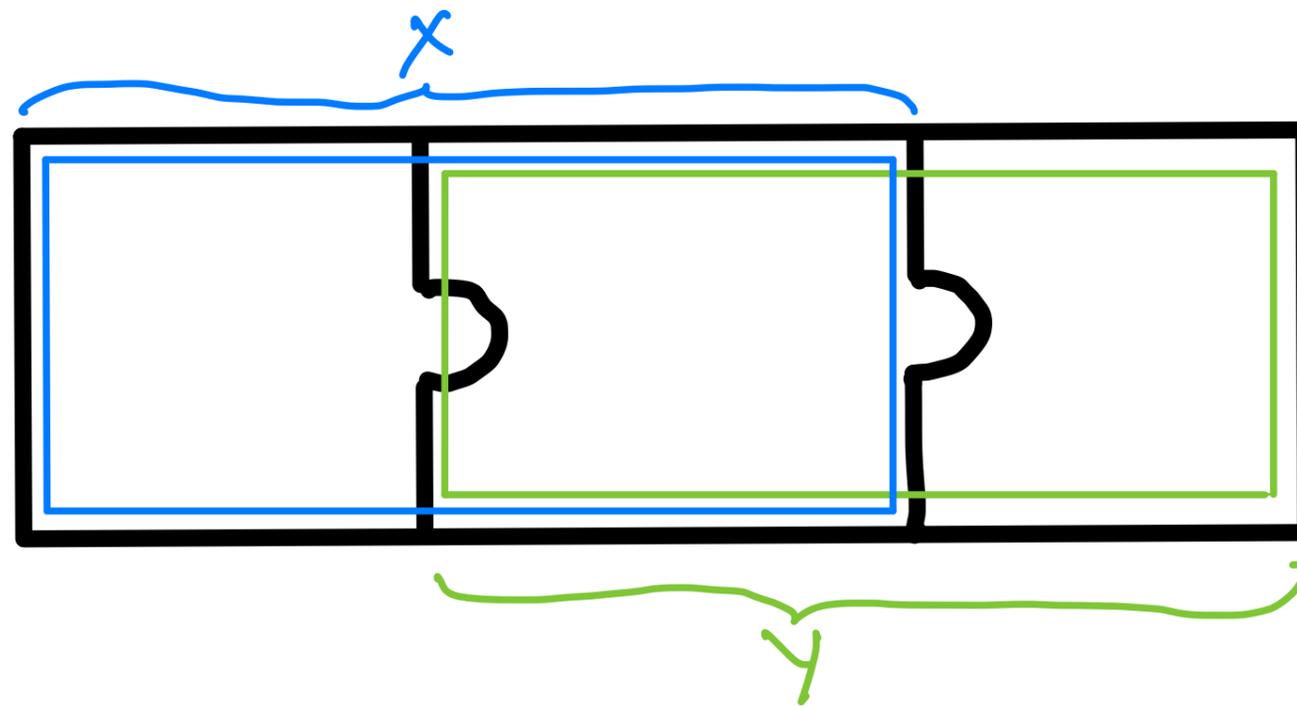
Optical illusion:

You might see this as an identification of a quotient group of X with a subgroup of Y. But that's the same thing, isn't it?

Homework: In the picture, label $\ker \phi$ and $\operatorname{im} \phi$.

Notice the first isomorphism theorem $X/\ker \phi \cong \operatorname{im} \phi$ in what you did.

Important Feature / Caution:

There is no object that is the "union" of X and Y (corresponding to all three puzzle pieces at once).

The only shapes you are allowed to discuss are parts (subs and quotients) of those shown (X and Y).

To make this clearer, notice the difference between what is depicted in the two pictures we have just discussed.
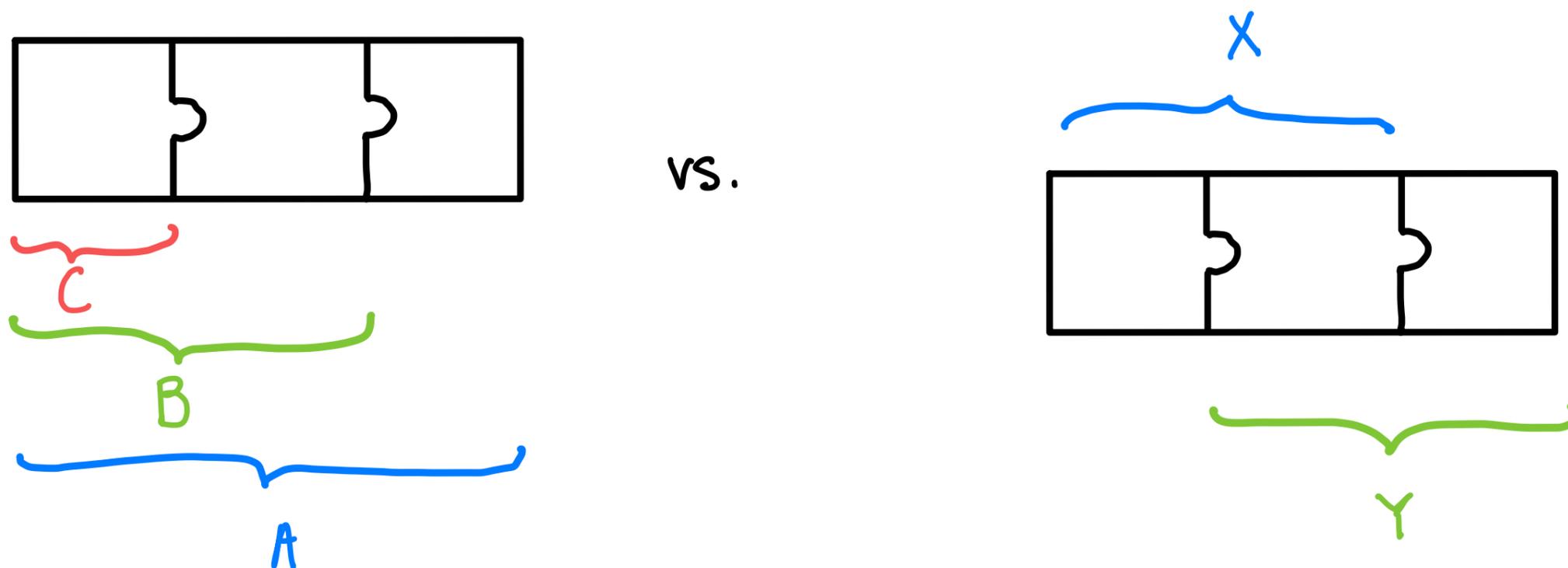
A priori, the information sounds quite different —

a length three filtration   vs.   a homomorphism.

But I now see that the only real difference is that in the second picture, we are not asserting the existence of an abelian group corresponding to the union of all three pieces.

So for example, we can immediately take a length three filtration and obtain a homomorphism of abelian groups, and we see precisely what information we have lost by this change in point of view.

Here is a picture of two morphisms $A \underset{C}{\overset{B}{<}}$
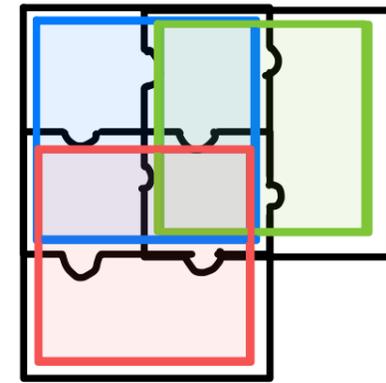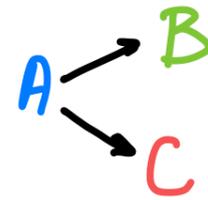
Question: Can you see $B \Big/ \Big(\text{im} \big(\ker (A \to C) \longrightarrow B \big) \Big)$?

Notice:

I can't draw a picture of a triangle $A \underset{C}{\overset{B}{<}}{\downarrow}$ that doesn't commute because I cannot draw two different maps $A \to C$ in this way.
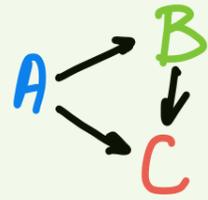
But I can draw a **commuting triangle**.

Here is a composition of two morphisms $A \to B \to C$,

or equivalently, a commuting diagram

$$A \longrightarrow B$$
$$\searrow \quad \downarrow$$
$$C$$
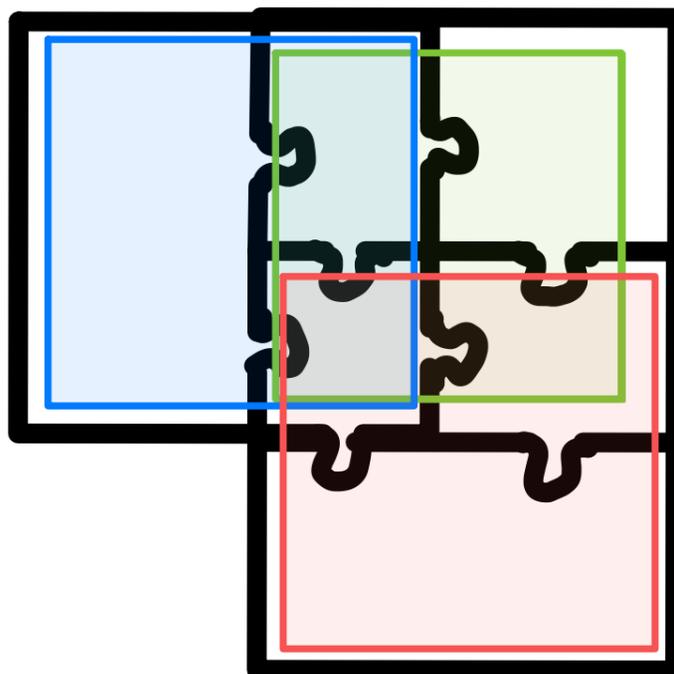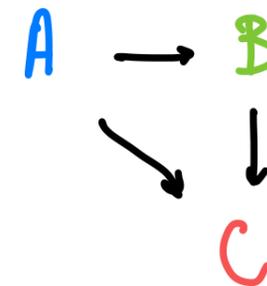
Why did I draw it this way?

Why not more like a traditional Venn diagram?:
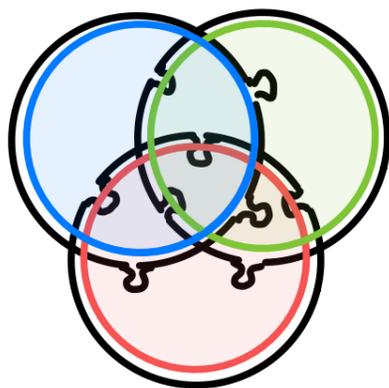
It isn't because the regions are depicted as circles — that isn't relevant.

(One clue: count the small regions.)

Notice that this picture needs to be "two-dimensional", not linear.

Homework:

Can you quickly describe each of the smallest regions in terms of A, B, and C, using kernels and cokernels of the maps in the commutative diagram? (First you will have to give them names, of course.)

Whenever you see a commuting triangle, you should forevermore "see" this picture. (and you will immediately see all the relevant subquotients at once).

---

commuting triangle.
Whenever you see this picture, you should immediately "see" a

Homework: How would you draw $A \to B \to C$ if you knew further that it is a complex, i.e. that the composition $A \to C$ is zero?

what if furthermore it were exact, i.e. that

$$\ker(B \to C) = \operatorname{im}(A \to B) \ ?$$

(No fair flipping ahead!)

$$A \rightarrow B$$
$$\downarrow \quad \downarrow$$
$$D \rightarrow C$$
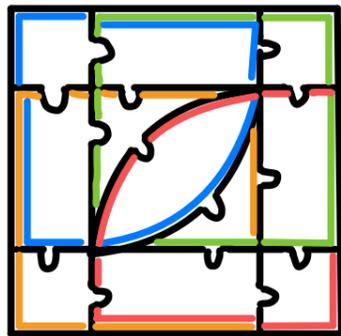
or, if you prefer,
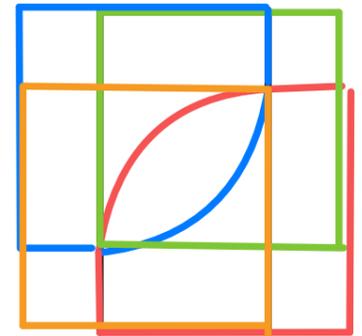
(This was quite tricky to think through!)

Happily, this is still a picture we can draw in two dimensions!

Whenever you see a commuting square, you should "see" this picture. (and you will immediately see all the relevant subquotients at once).

We have already drawn a

A

B

$C = A/B$
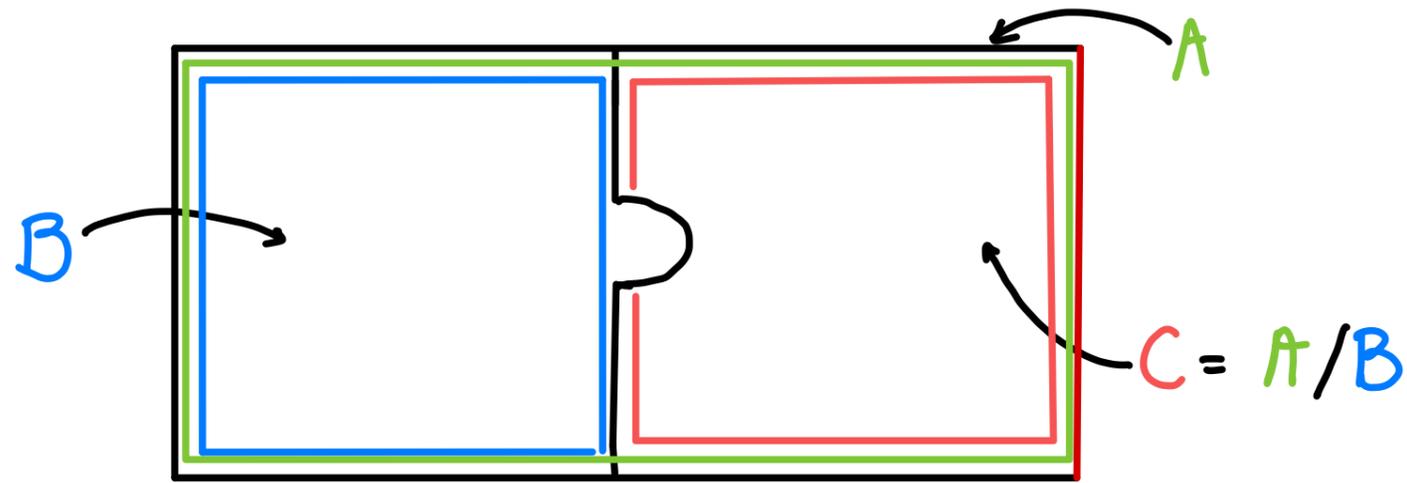
$$0 \longrightarrow B \longrightarrow A \longrightarrow C \longrightarrow 0$$

Here is a longer exact sequence:

$\cdots$

C^{i-1}

C^{i+1}

$\cdots$

$\cdots$

C^{i}

C^{i+2}

$\cdots$
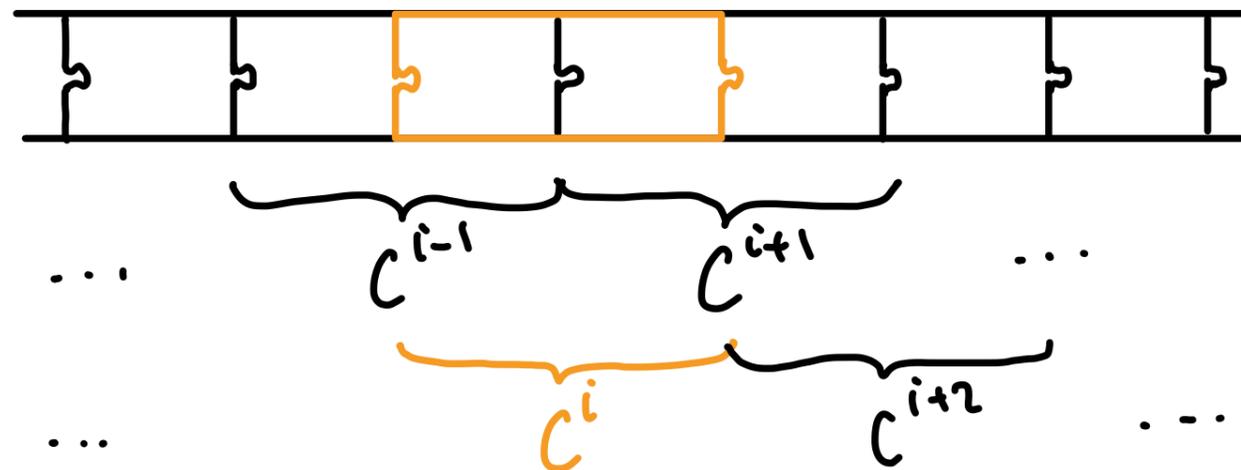
Perhaps I should have said this in the opposite order:

Given an exact sequence

$$\to \cdots \to C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} C^{i+2} \to \cdots$$

I begin to visualize it as:

$$\cdots \to \boxed{C^{i-1}} \to \boxed{C^i} \to \boxed{C^{i+1}} \to \boxed{C^{i+2}} \to \cdots$$

I then put the pieces together (using overlaps to encode relationships):
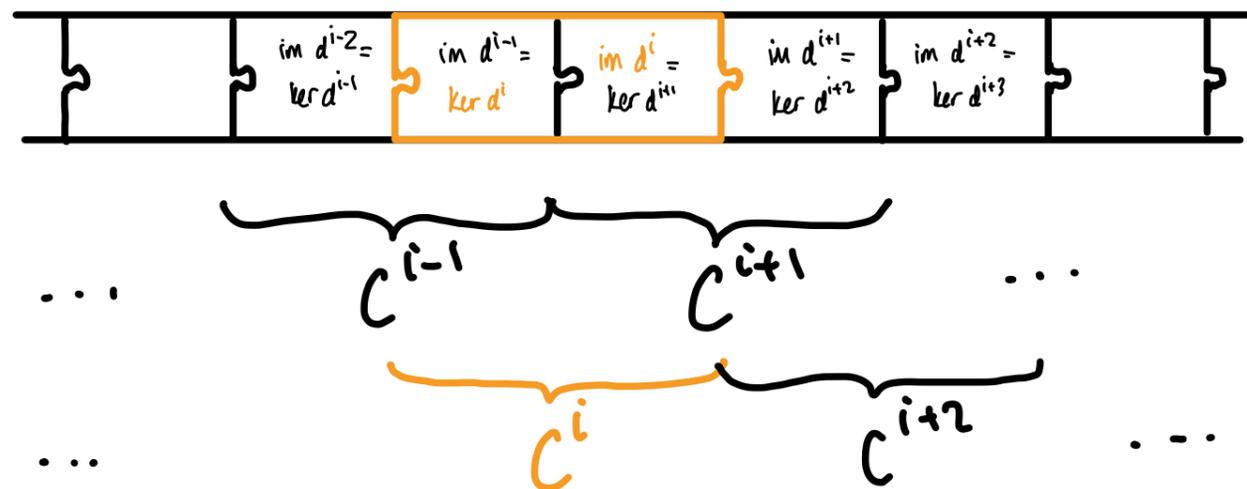
Here is the snake lemma, in a tidy little package:

In more detail:

$$0 \to A \to B \to C \to 0$$
$$ \downarrow \downarrow \downarrow$$
$$0 \to D \to E \to F \to 0$$
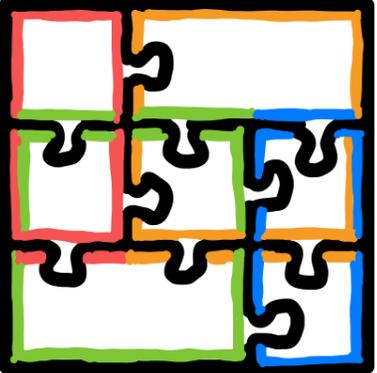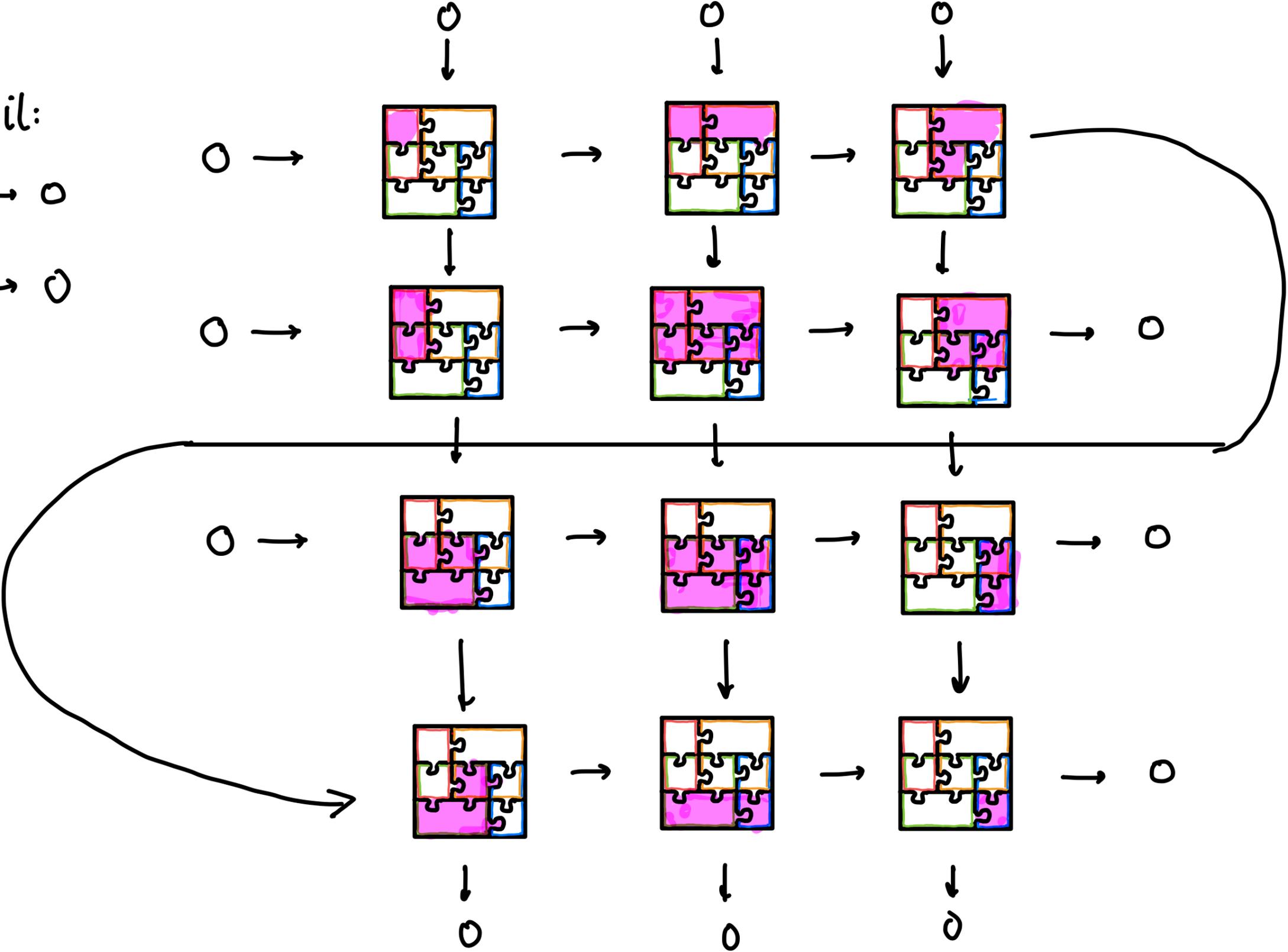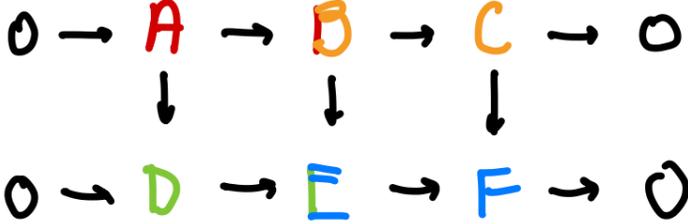
You can turn this into a formal algebraic proof of the snake lemma, by defining what each piece means, and showing that they have the relationships that the pictures depict.

You might begin:
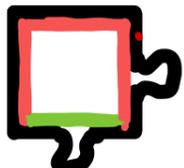
Define $:= \ker(A \to D)$
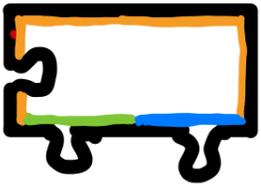
$:= \ker(B \to E) / \ker(A \to D)$ (after showing $\ker(A \to D)$ is a subgroup of $\ker(B \to E)$)

and continuing. It isn't vastly simpler than just proving the snake lemma "traditionally" but it puts everything into perspective. For example, you will recognize the well-camouflaged piece   is really the star of the show.

Let us now draw a complex:

$$\to \cdots \to C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} C^{i+2} \to \cdots$$

I begin to visualize it as:

$$\cdots \to \boxed{C^{i-1}} \to \boxed{C^i} \to \boxed{C^{i+1}} \to \boxed{C^{i+2}} \to \cdots$$

and I put the pieces together (overlapping a bit less than before) to get

I can't help but give names to what I see:

$$\underbrace{\text{coker } d^{i-2}}\ \underbrace{\text{coker } d^{i-1}}\ \underbrace{\text{coker } d^i}\ \underbrace{\text{coker } d^{i+1}}$$

$$\underbrace{\text{ker } d^{i-1}}\ \underbrace{\text{ker } d^i}\ \underbrace{\text{ker } d^{i+1}}\ \underbrace{\text{ker } d^{i+2}}$$

$$\ldots \qquad \text{im}^{i-2}\ H^{i-1}\ \text{im}^{i-1}\ H^i\ \text{im}^i\ H^{i+1}\ \text{im}^{i+1}\ H^{i+2}\ \text{im}^{i+2} \qquad \ldots$$

$$\ldots \qquad \underbrace{C^{i-1}} \qquad \underbrace{C^{i+1}} \qquad \ldots$$

$$\ldots \qquad \underbrace{C^i} \qquad \underbrace{C^{i+2}} \qquad \ldots$$
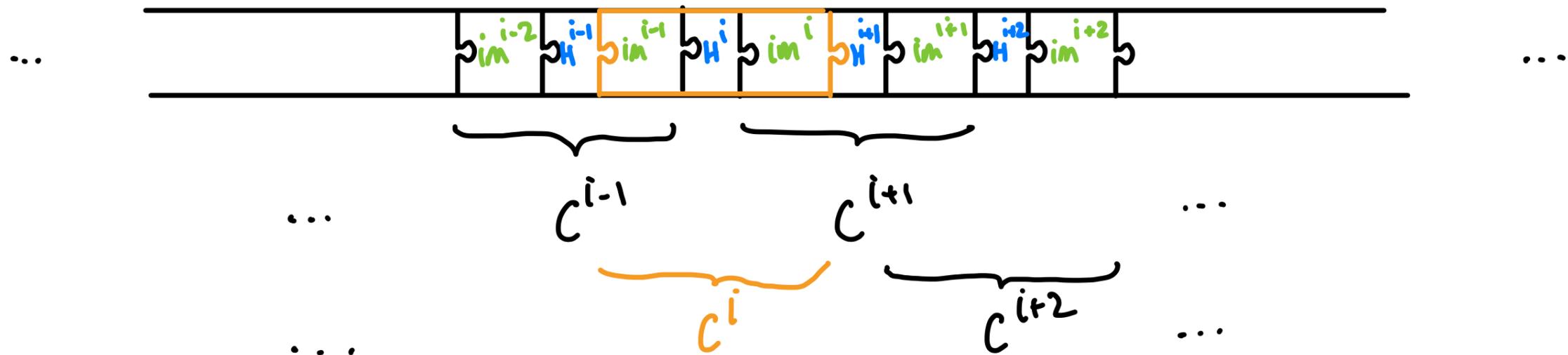
We have accidentally discovered cohomology as well as ...

We see the important short exact sequence

$$0 \to \operatorname{im} d^{i-1} \to \ker d^i \to H^i \to 0$$

(the traditional **definition** of H^i)

but also

$$0 \to H^i \to \operatorname{coker} d^{i-1} \to \operatorname{im} d^i \to 0$$

(its equally important but much-neglected "mirror image").

We also have up front in our consciousness:

$$0 \to \ker d^i \to C^i \to \operatorname{im} d^i \to 0$$

$$0 \to \operatorname{im} d^{i-1} \to C^i \to \operatorname{coker} d^{i-1} \to 0.$$

(Trick question: we see the kernel and the cokernel.
We see the image. Where is the coimage?)

Until now, I never realized that every complex gives a
"kernel-cokernel" exact sequence:

$$\dots \longrightarrow \ker d^{i-1} \longrightarrow \operatorname{coker} d^{i-2} \longrightarrow \ker d^{i} \longrightarrow \operatorname{coker} d^{i-1} \longrightarrow \ker d^{i+1} \longrightarrow \dots$$

which doesn't seem immediately useful, but certainly brightens my day!

If we have a morphism of complexes, there is an induced morphism of cohomology groups. We can show this by writing down algebra and shuffling symbols. But it helps to look at this picture first.

$$C^{n-1} \to C^n \to C^{n+1}$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$D^{n-1} \to D^n \to D^{n+1}$$
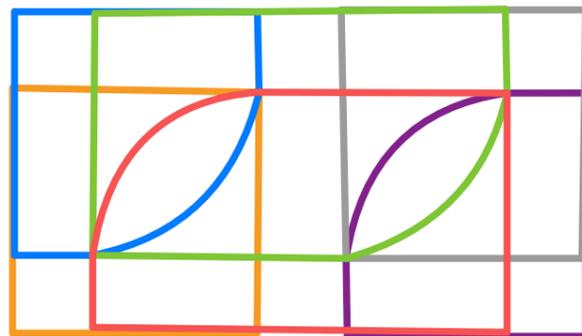
(How to build it: notice the two commuting squares and also the two three-term complexes.)

The puzzle connectors are omitted for aesthetic reasons, but the "arrows" always go "right" and "down". Here is the "puzzle picture".
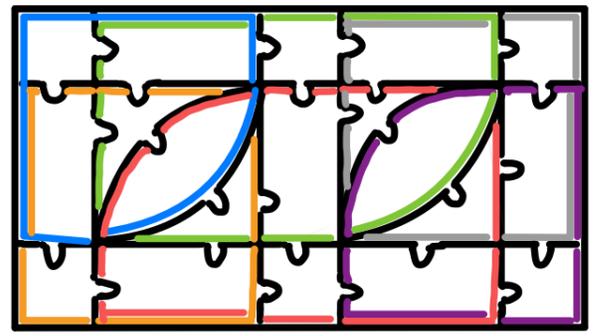
We immediately see the map of cohomology:

given by

Notice how "small" the image of this map is!

Let us now derive the **long exact sequence in cohomology** arising from a short exact sequence of complexes:
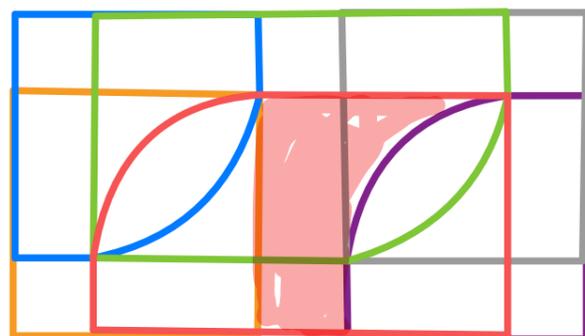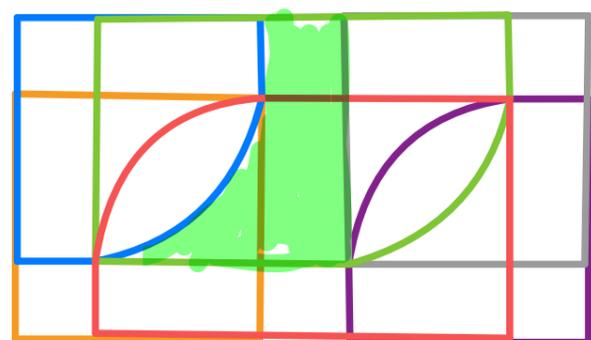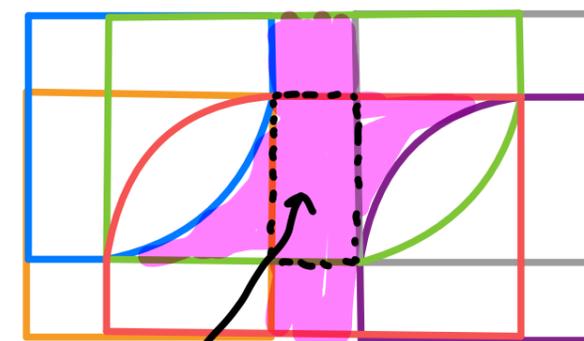
$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to & B^{n-2} \to & B^{n-1} \to & B^n \to & B^{n+1} \to & B^{n+2} \to \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to & C^{n-2} \to & C^{n-1} \to & C^n \to & C^{n+1} \to & C^{n+2} \to \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to & D^{n-2} \to & D^{n-1} \to & D^n \to & D^{n+1} \to & D^{n+2} \to \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

we do this by starting with the complex for the C's
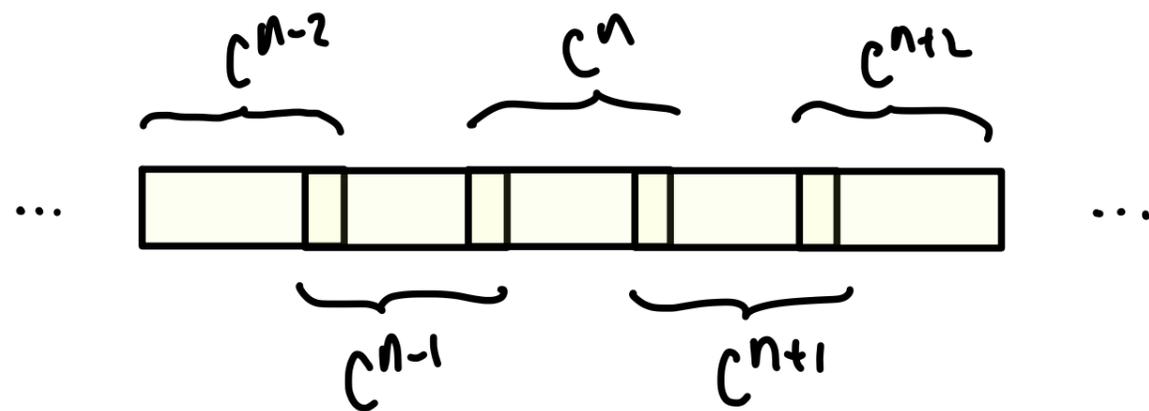
We then note that each C^n has a (two-term) filtration

$0 \to B^n \to C^n \to D^n \to 0$ so that the differential

$d^n : C^n \to C^{n+1}$ maps B^n to B^{n+1}, thereby ensuring that

the $B^\bullet$ and the $D^\bullet$ sequences are complexes.

You might want to depict the filtration like this:
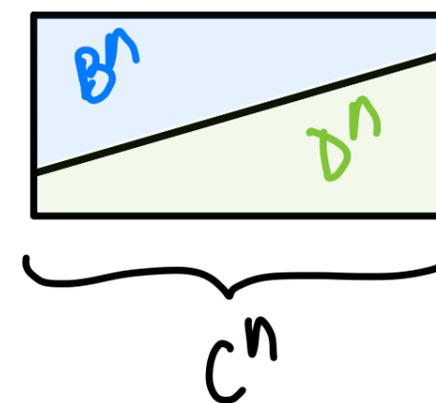
so the short exact sequence of complexes looks like this:
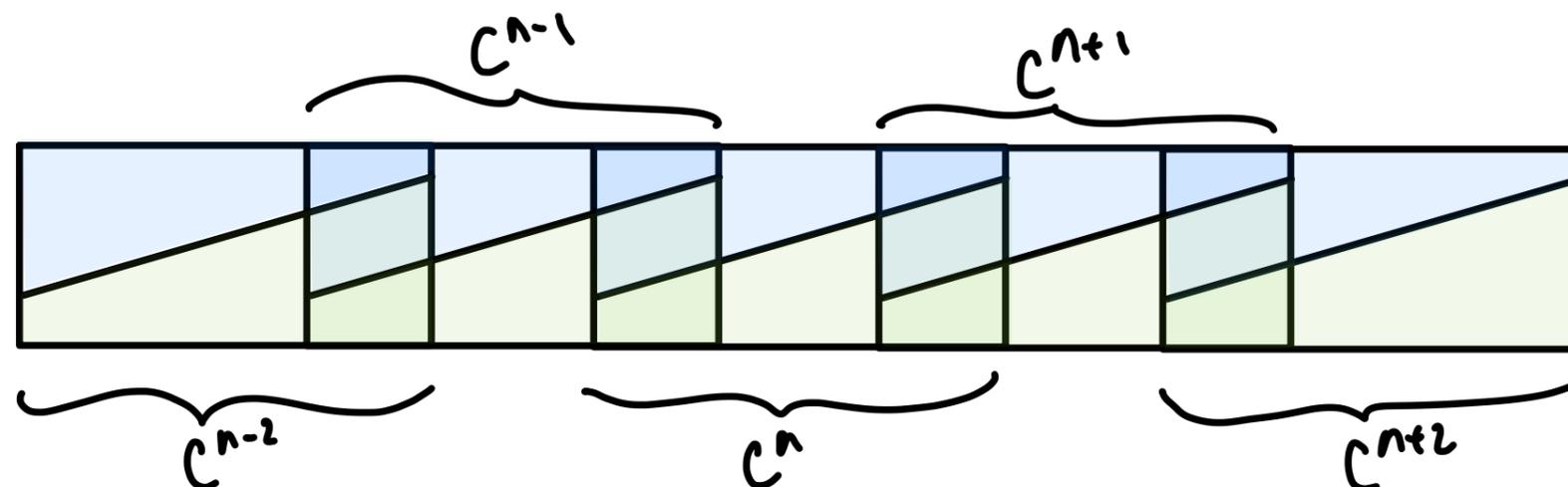
(Notice the snake lemma picture in it, sideways.)

To make sure we are getting across the "subs" and "quotients", we add in the puzzle connectors.

$$C^{n-1}$$

$$C^{n+1}$$

$$C^{n-2}$$

$$C^{n}$$

$$C^{n+2}$$
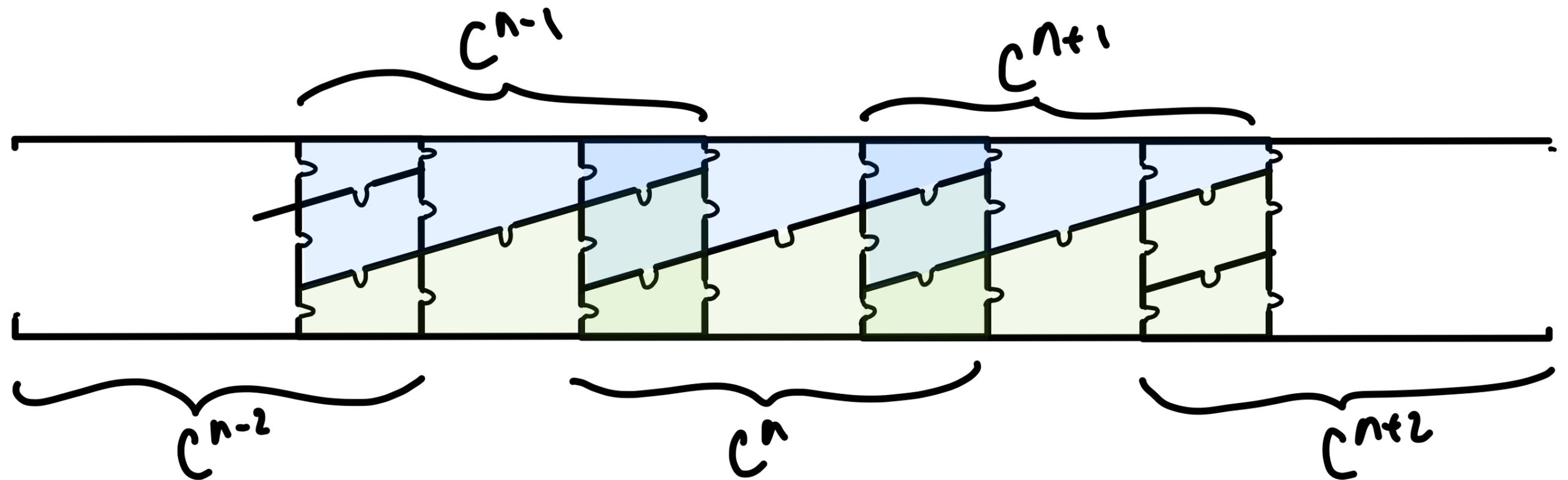
The long exact sequence in cohomology now becomes visually clear:

Here is our short exact sequence of complexes, with the cohomology groups of each highlighted:
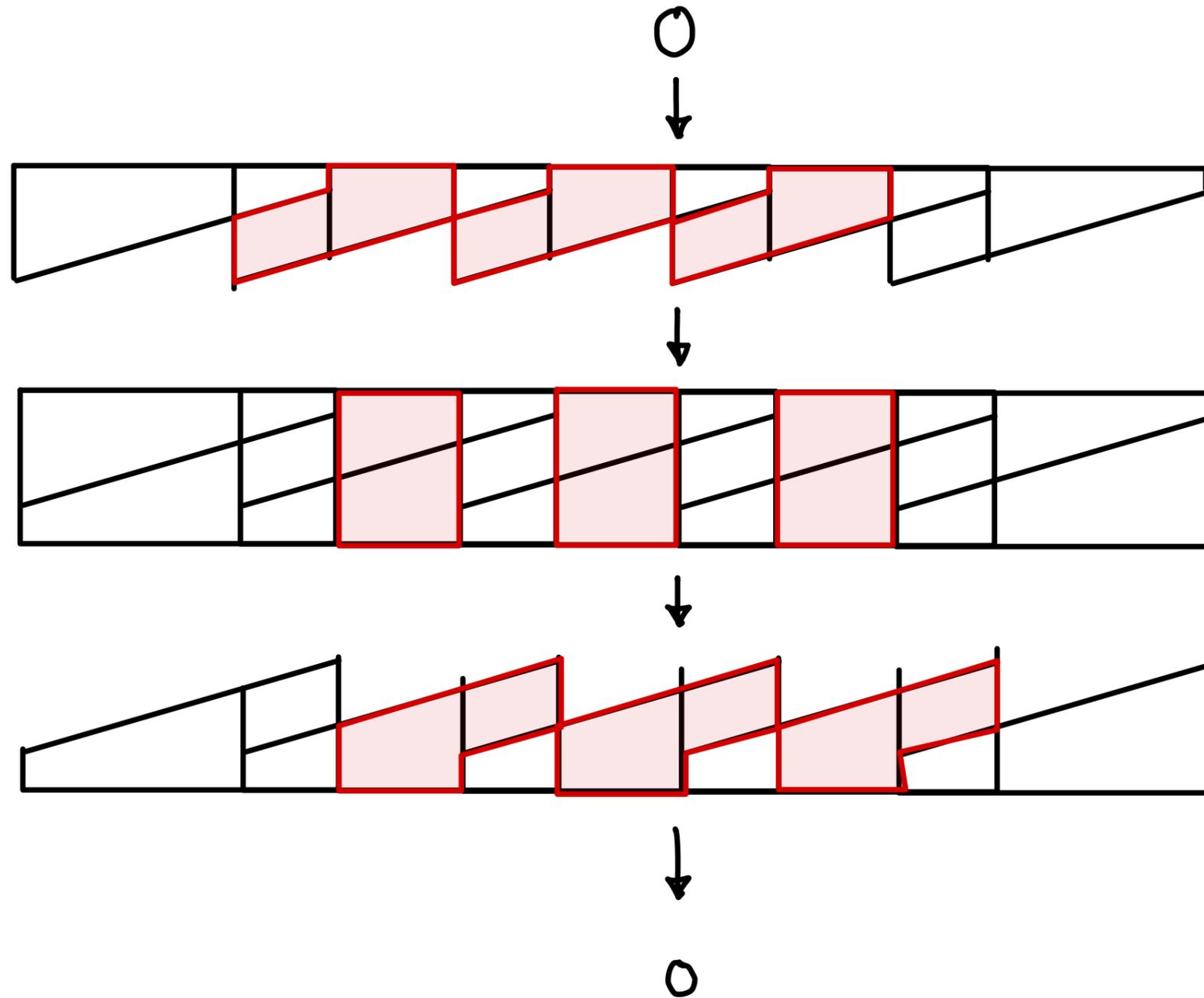
The maps on cohomology now visually/pictorally give you our desired long exact sequence. You might now prefer it depicted thus:

C^{n-1}

C^{n+1}

C^{n-2}

C^{n}

C^{n+2}
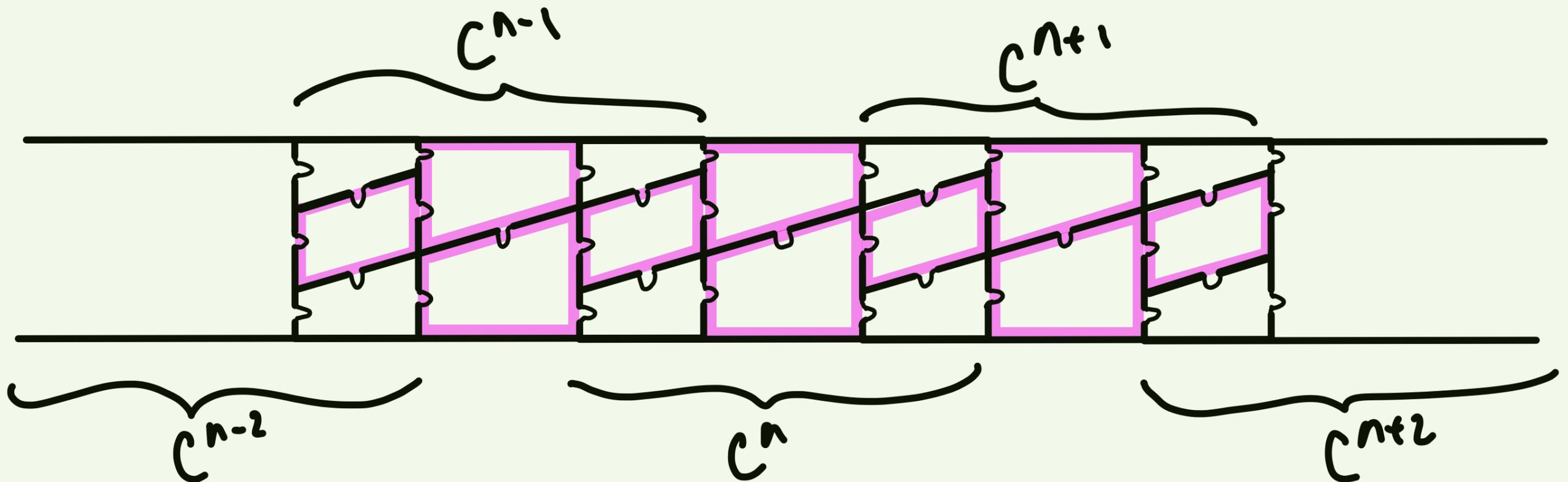
Pictoral proof/construction of the long exact sequence in cohomology arising from a short exact sequence of complexes.

To turn this into a proof that others can understand, you need only turn the following into algebra, and prove them:
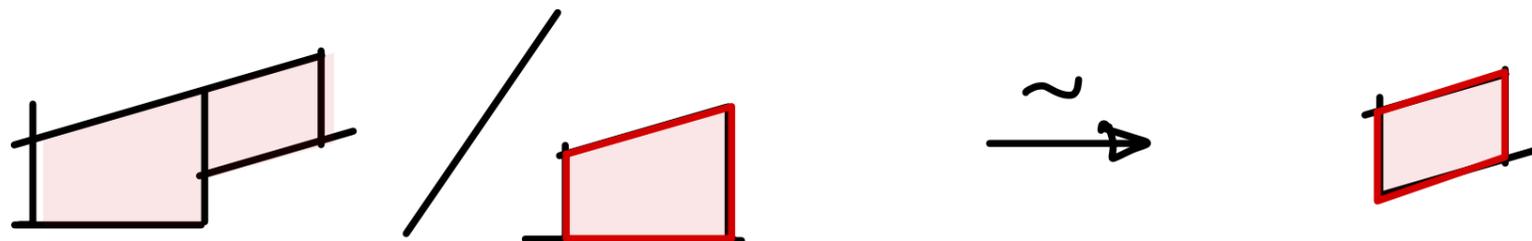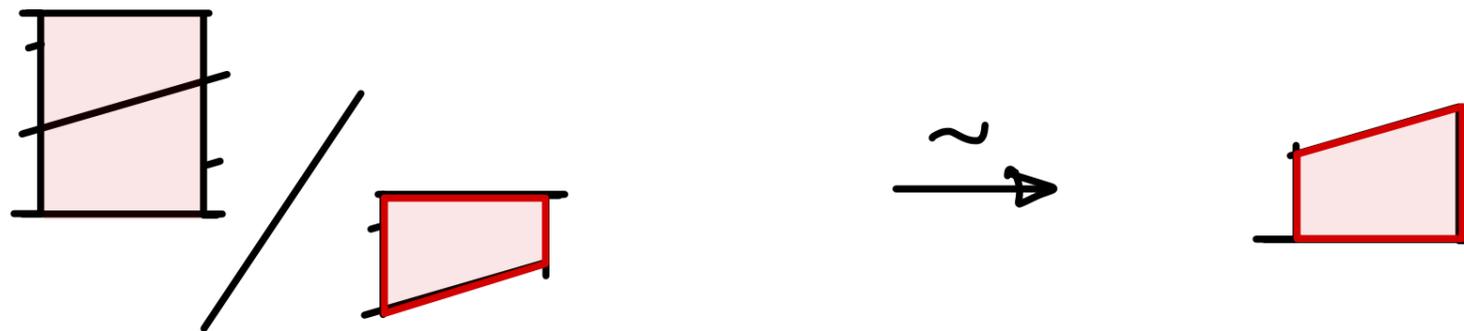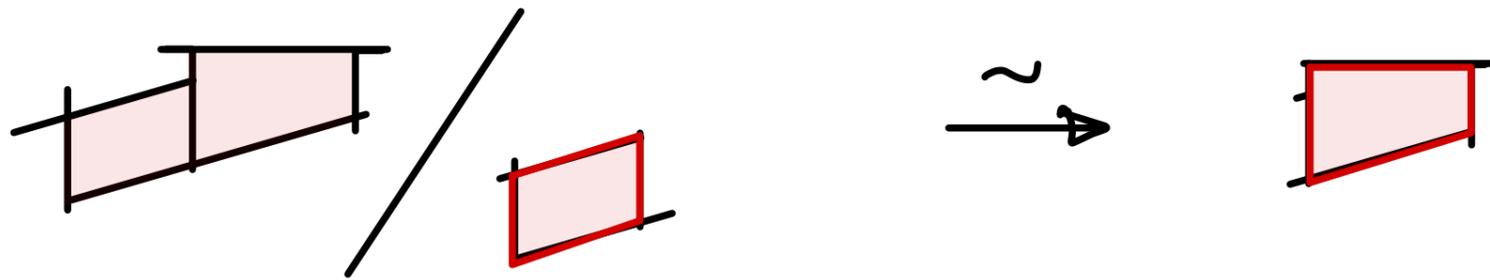
This is a generalization of the long exact sequence in cohomology, but will be more complicated.

Suppose we have a complex

$$C^{n-2} \qquad C^n \qquad C^{n+2}$$

$$C^{n-1} \qquad C^{n+1}$$
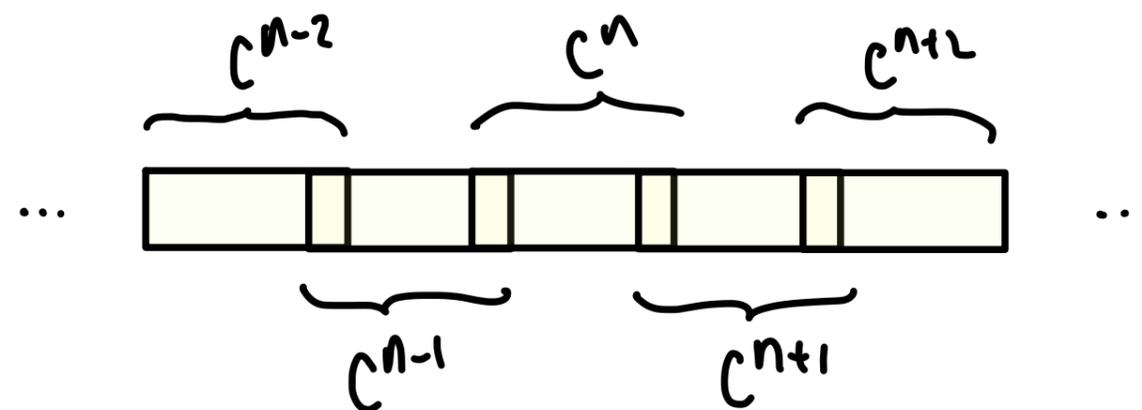
and the complex is "filtered". For simplicity, let us take a four-step filtration. You will see how the general case works from this. Having a four step filtration means that we have a filtration

$$0 \subset F^0 C^n \subset F^1 C^n \subset F^2 C^n \subset F^3 C^n \subset F^4 C^n = C^n$$

We will clearly want to picture this as something like:

F^1
$F^2\{$
F^3 $\}$
F^4
C^n
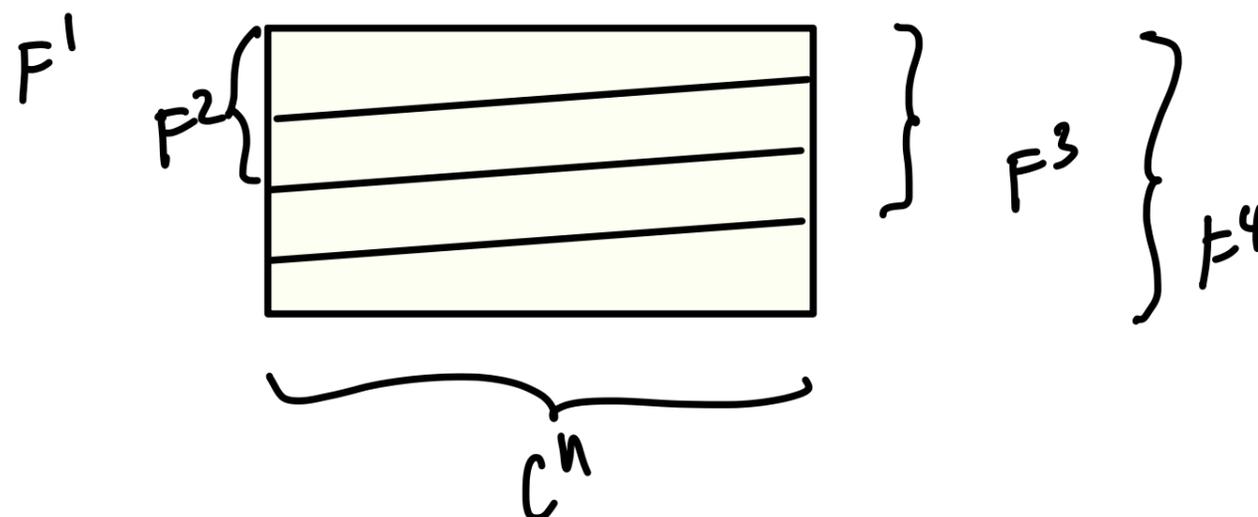
We want to draw this so that $\boxed{d^n(F^j C^n) \subset F^j C^{n+1}}$

but we need to keep track of the ways in which $d^n(F^j C^n)$ meets $F^k C^n$ for $k < n$ as well.

You may want to ponder this a bit before seeing how I do it.

It turns out that the right way to draw the filtration on C^n is something essentially like this:

$$F^4 C^n = C^n$$

$$F^3 C^n$$

$$F^2 C^n$$

$$F^1 C^n$$

or

$$F^1 C^n$$

$$F^2 C^n$$

$$F^3 C^n$$

$$F^4 C^n = C^n$$
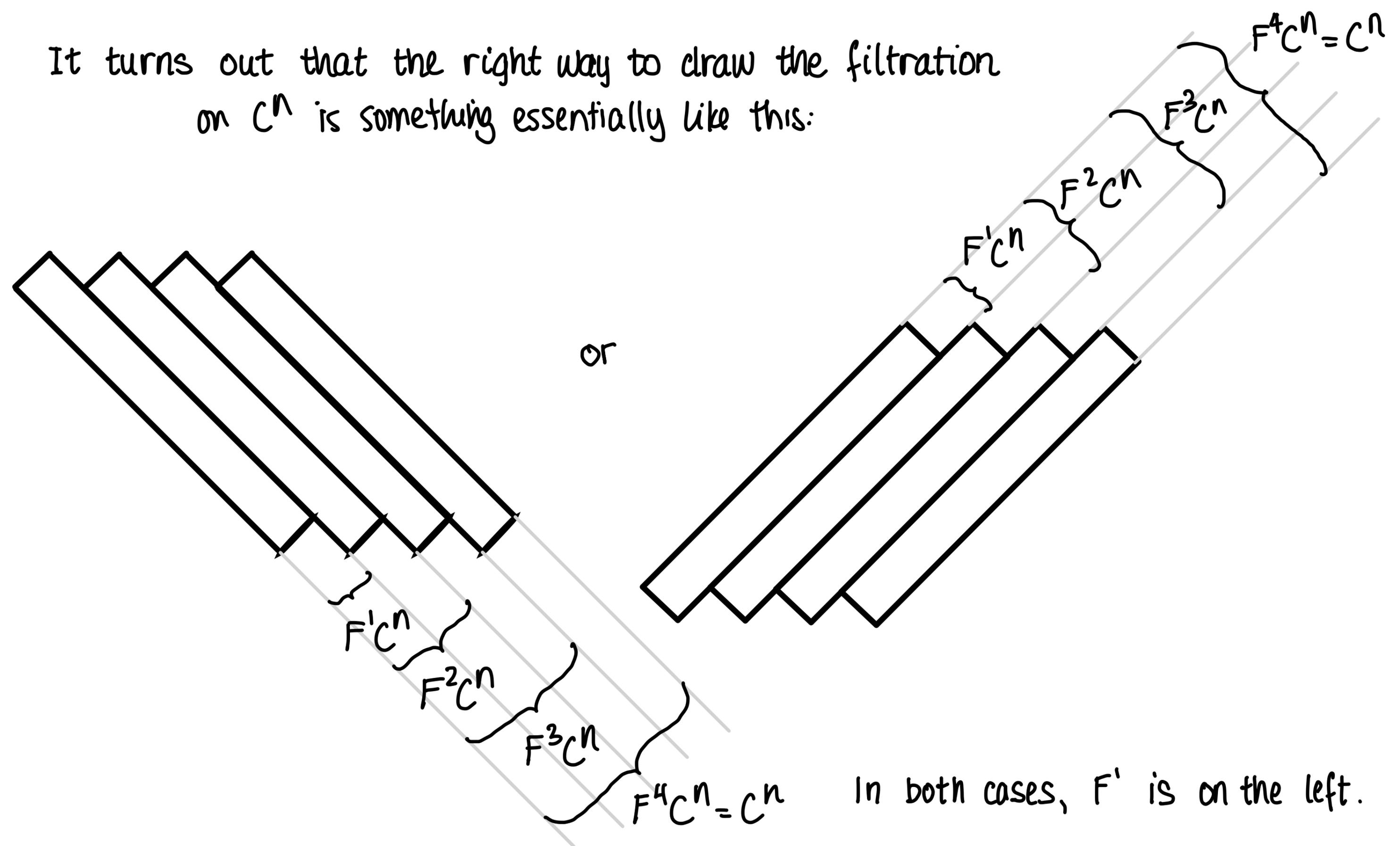
In both cases, F^1 is on the left.

In general, "subs" are on the left and "quotients" are on the right.

Here then is what the filtered complex looks like.
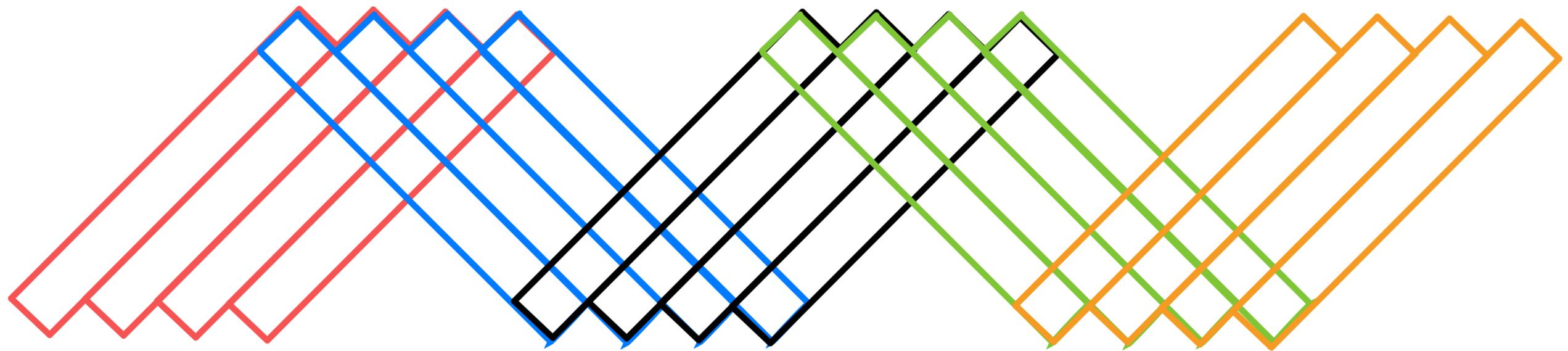
$$C^{n-2} \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} C^{n+2}$$

Here are three cohomology groups, highlighted, corresponding to the part (subquotient) of C^n not overlapping with C^{n-1} or C^{n+1}

They come with an obvious filtration.

$F^1 H^n$ $F^2 H^n / F^1 H^n$

$F^3 H^n / F^2 H^n$ $F^4 H^n / F^3 H^n$
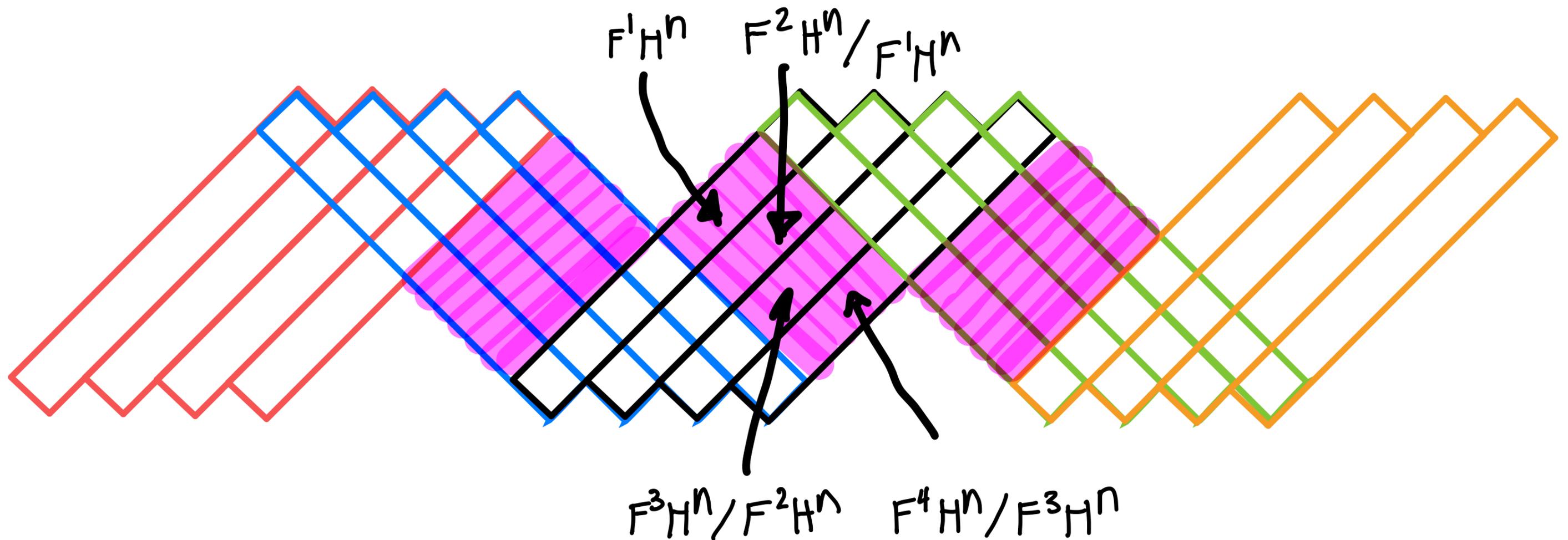
The "output" of the spectral sequence process will be the "parts" of the filtration $F^j H^n / F^{j-1} H^n$.

with the "subquotient" complexes, $F^j C^\bullet / F^{j-1} C^\bullet$.

This is "page 0" of the spectral sequence (E_0). Here they are, pictorially.

$F^1 C^\bullet$

$F^2 C^\bullet / F^1 C^\bullet$

$F^3 C^\bullet / F^2 C^\bullet$

$F^4 C^\bullet / F^3 C^\bullet$
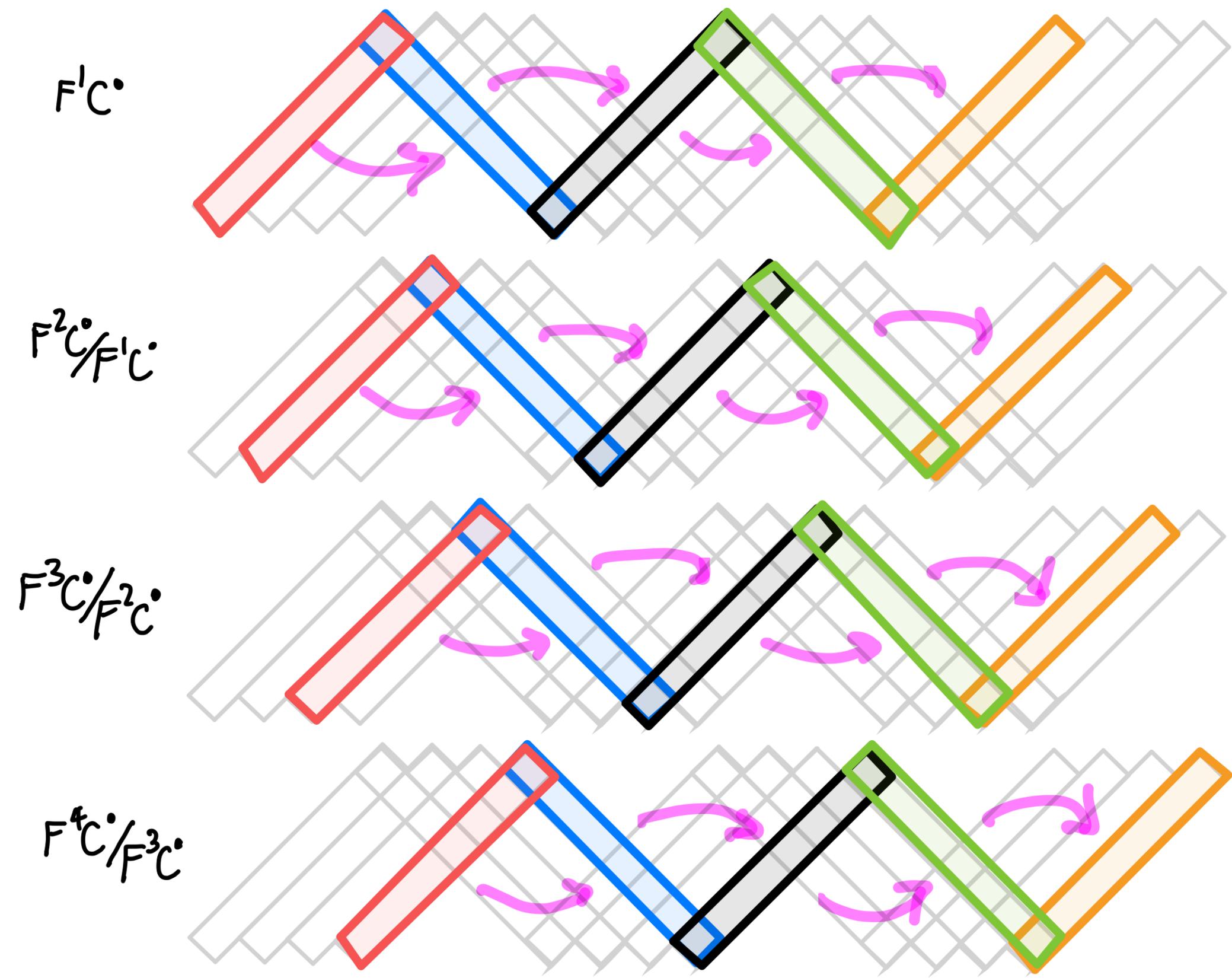
The obvious maps between the spaces (indicated by arrows) clearly describe a series of complexes. We take the cohomology of these complexes to obtain the first page of the spectral sequence, E_1.

Here is the first page E_1, with the entries displayed in a similar way.

We now have new natural "diagonal" morphisms, clearly (pictorially) yielding complexes.

$F^1C^\bullet$

$F^2C^\bullet/F^1C^\bullet$

$F^3C^\bullet/F^2C^\bullet$

$F^4C^\bullet/F^3C^\bullet$
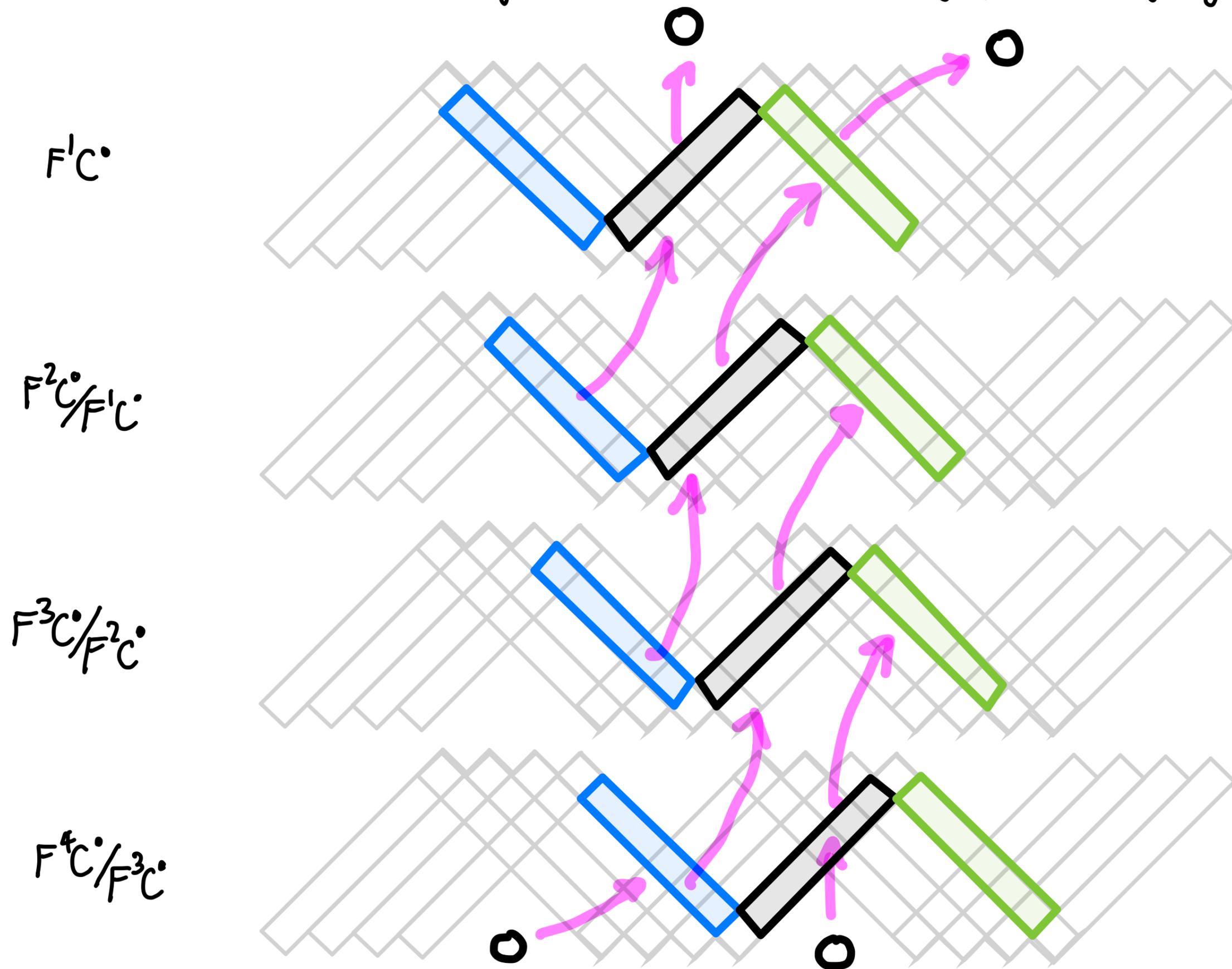
We now take cohomology of these complexes, obtaining the second page E_2 of the sequence. Now new arrows have revealed themselves.
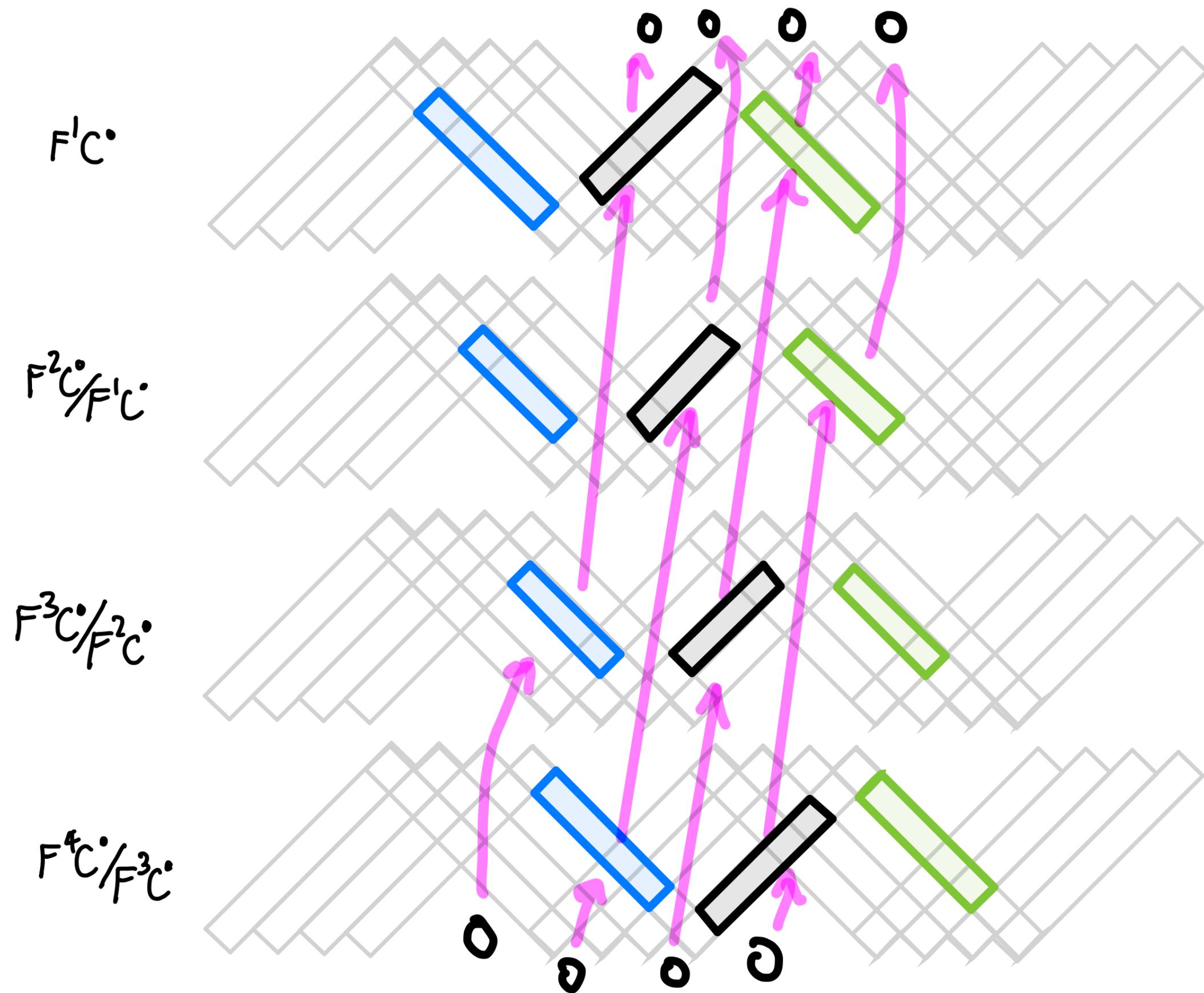
We now take cohomology of <u>these</u> complexes, obtaining the third page E_3 of the sequence. Now <u>new</u> arrows have revealed themselves.

$F^1 C^\bullet$

$F^2 C^\bullet / F^1 C^\bullet$

$F^3 C^\bullet / F^2 C^\bullet$

$F^4 C^\bullet / F^3 C^\bullet$
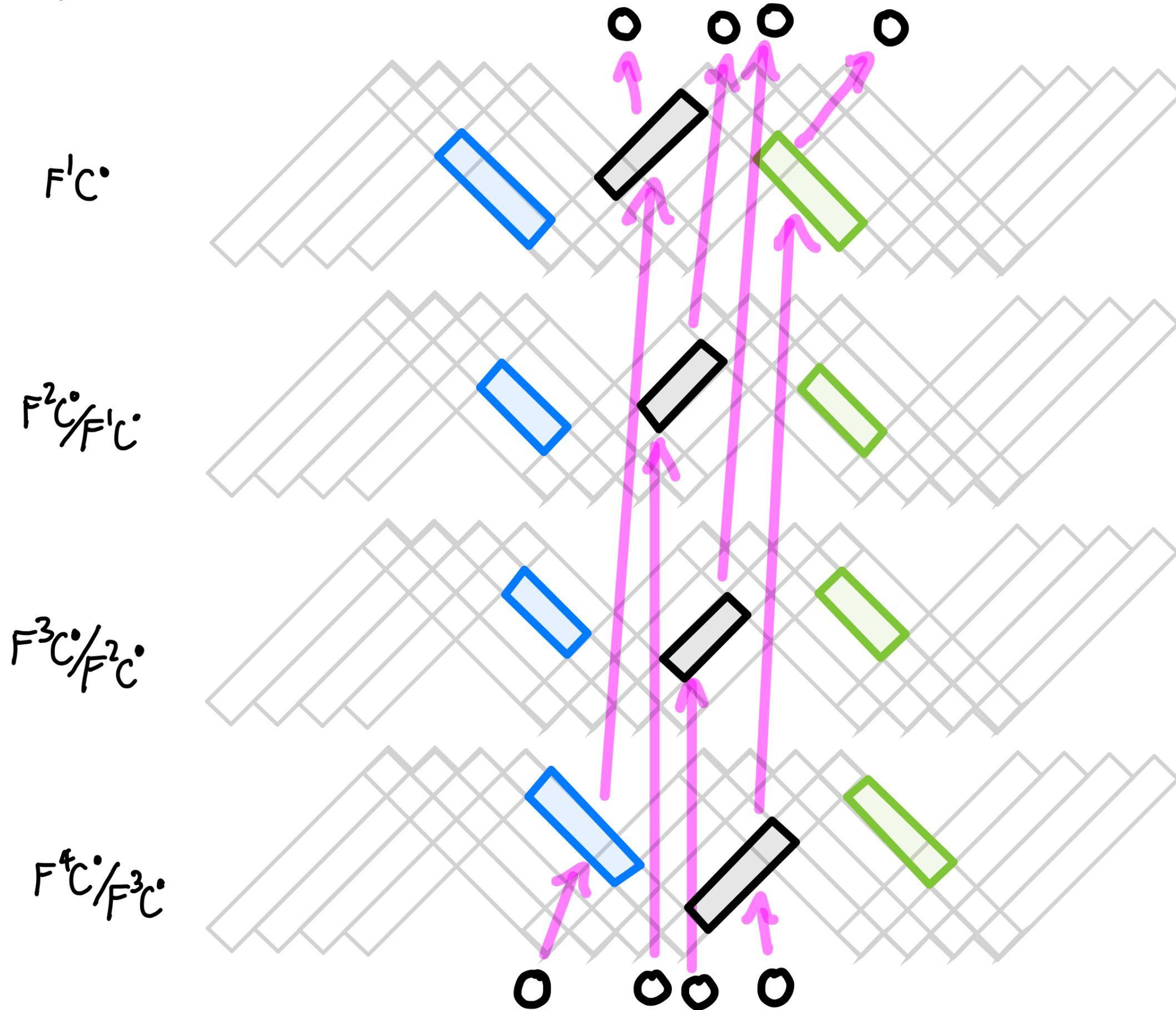
Two of our four pieces of $H^i(C^\bullet)$ have already converged

We now take cohomology of *these* complexes, obtaining the final page E_4. We have found the desired pieces of $H^\bullet(C^\bullet)$!

$F^1 C^\bullet$

$F^2 C^\bullet / F^1 C^\bullet$

$F^3 C^\bullet / F^2 C^\bullet$

$F^4 C^\bullet / F^3 C^\bullet$

H^{n-1}

H^n

H^{n+1}
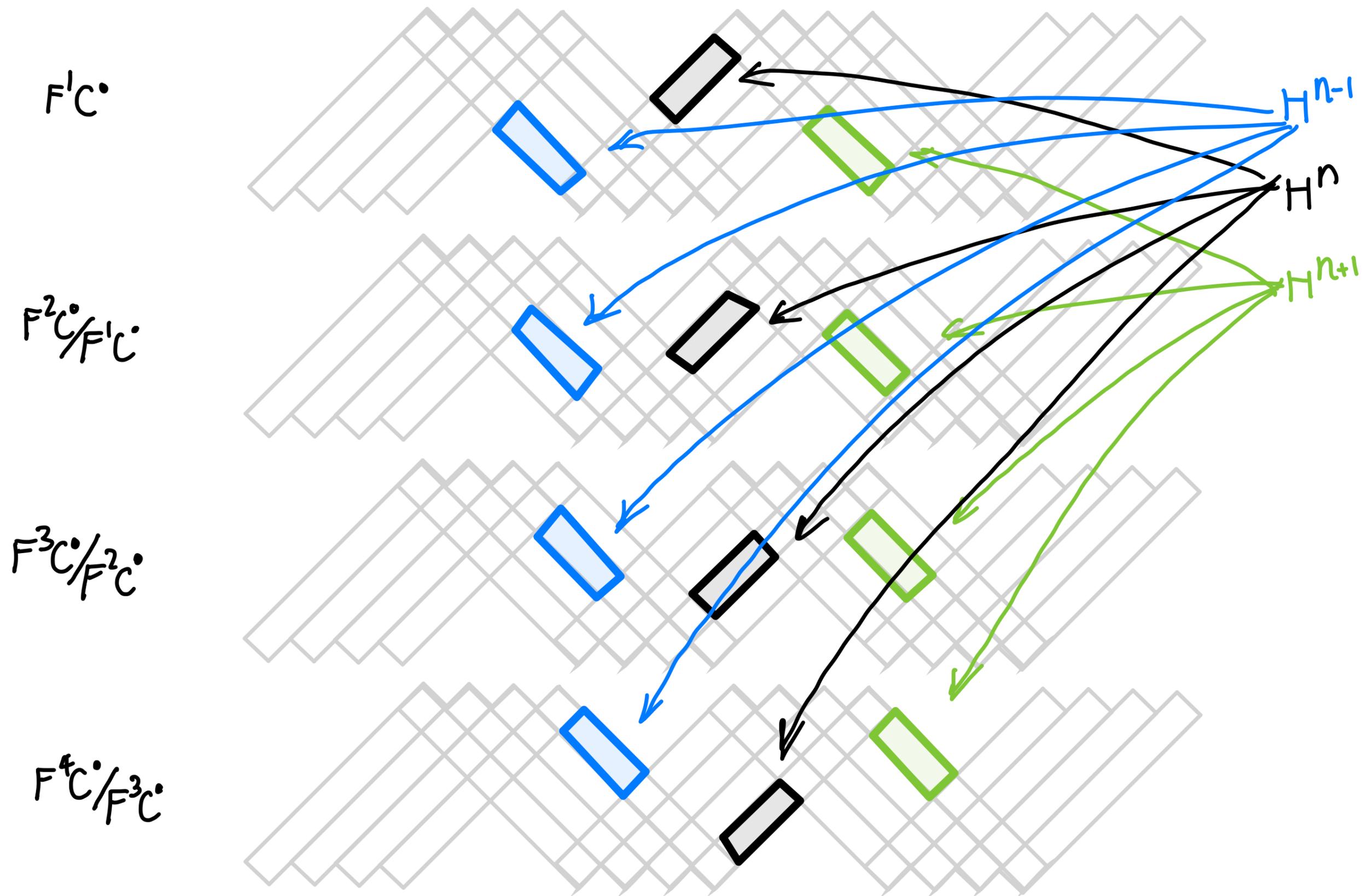
We label some of the regions as follows.
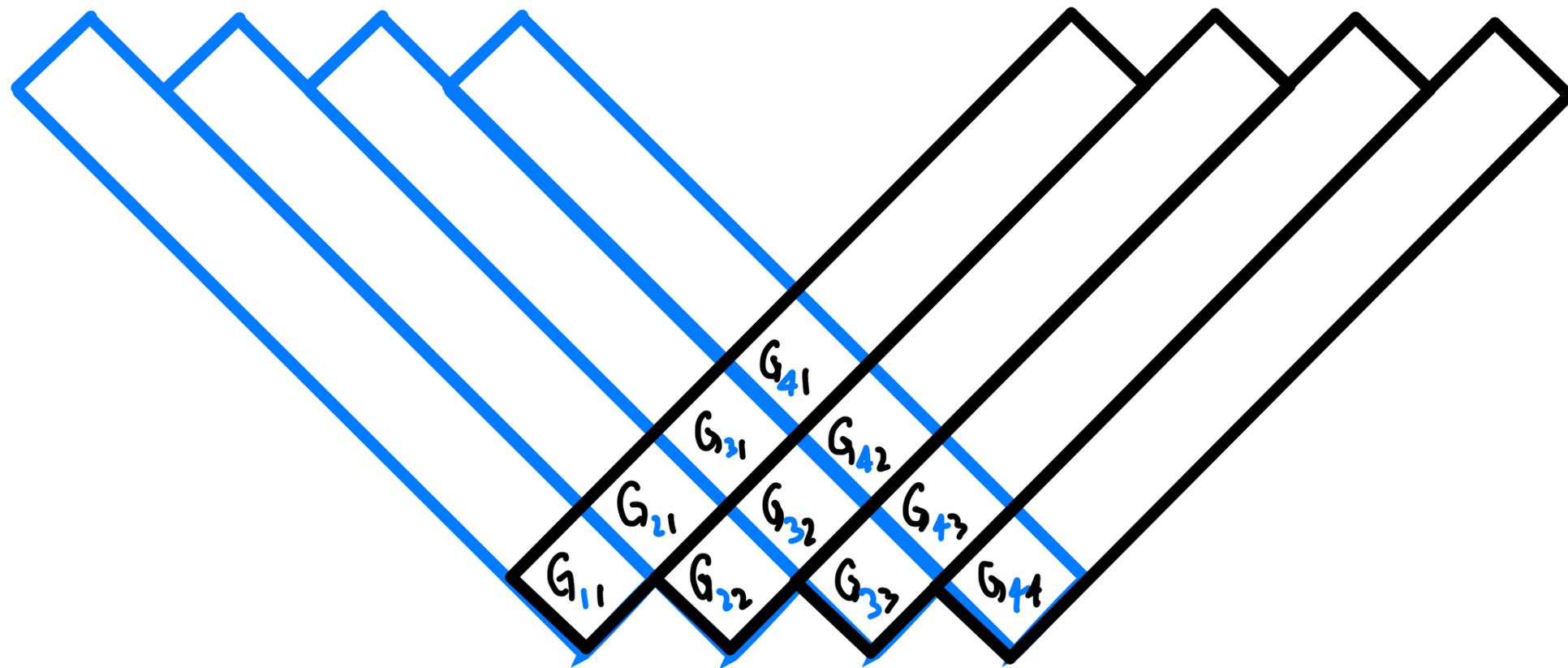
By the pth page E_p, we have surgically removed all the pieces G_{ij} with $i < j+p$.

You can now figure out how to turn this into a proof,

by predicting each entry on the pth page, observing that the desired

arrows exist, and then showing that we get complexes with the

desired cohomology.

Here is most of the proof...

You will see that you are forced to interpret/define:

$$\ker(C^n \to F^{q'-1}C^{n+1}) \cap F^p C^n$$

$$\overline{\mathrm{im}(F^{r'-1}C^{n-1}) \cap F^p C^n + \ker(C^n \to F^{q'-1}C^{n+1}) \cap F^{p-1}C^n}$$

Call this $A_{r \geq p \geq q}$

$F^p C^n$

$F^{r'}H^n$

$F^p H^n$

G_{rp} G'_{pq}

$\to$??? (Homework)

$F^{q'}H^n$

$$\ker(d^n) \cap F^{r'}C^n$$

$$\overline{\mathrm{im}(d^{n-1}) \cap F^{r'}C^n + \ker(d^n) \cap F^{q'-1}C^n}$$

Call this $B_{r' \geq q'}$
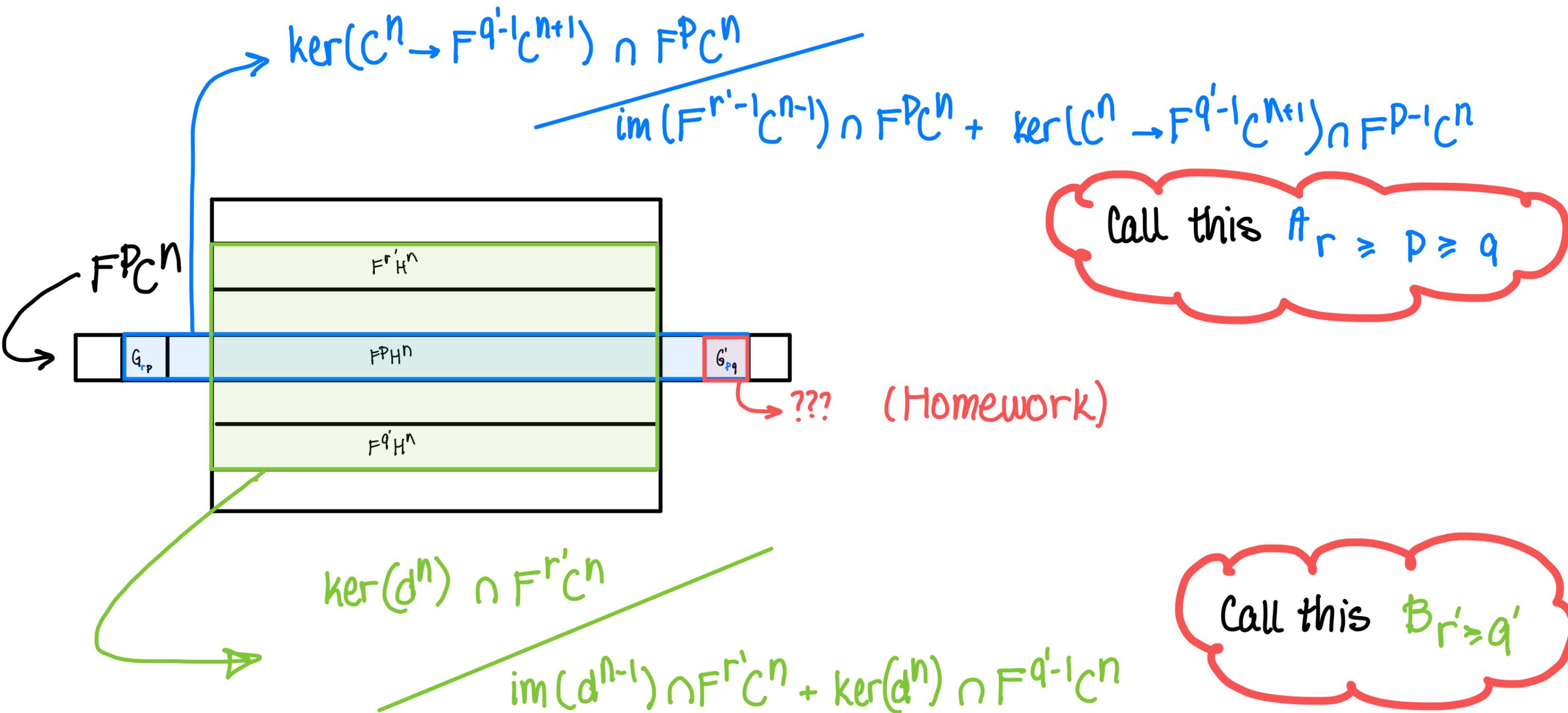
My point is that these crazy formulas no longer appear out of nowhere — they are just translations of very reasonable pictures.

Now describe each entry on each page of
your spectral sequence as some $A_{r \geqslant p \geqslant q}$

Then describe/prove three short exact sequences

$$0 \rightarrow G_{r,p} \rightarrow A_{r \geqslant p \geqslant q} \rightarrow A_{r-1 \geqslant p \geqslant q} \rightarrow 0$$

$\underbrace{\qquad}_{G_{r,p}} \underbrace{A_{r \geqslant p \geqslant q}}_{} \qquad A_{r-1 \geqslant p \geqslant q}$
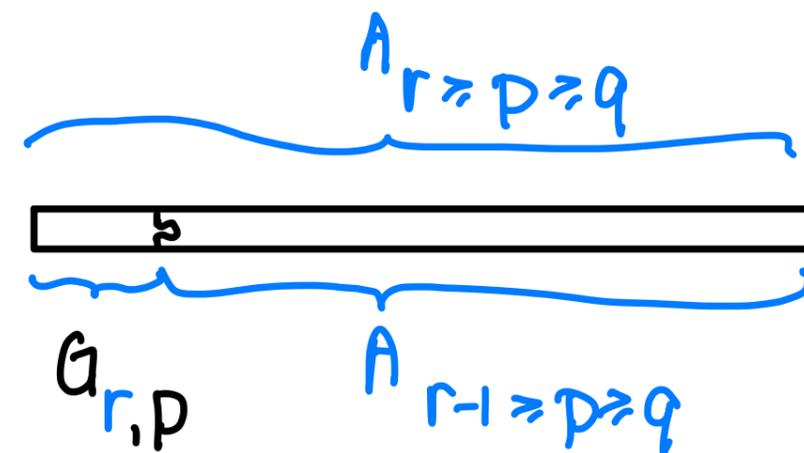
$$0 \rightarrow A_{r \geqslant p \geqslant q+1} \rightarrow A_{r \geqslant p \geqslant q} \rightarrow G'_{p,q} \rightarrow 0$$

$\underbrace{\qquad}_{A_{r \geqslant p \geqslant q+1}} \qquad G'_{p,q}$

$B_{r' \geqslant p+1}$

$B_{r' \geqslant q'}$

$B_{p \geqslant q}$
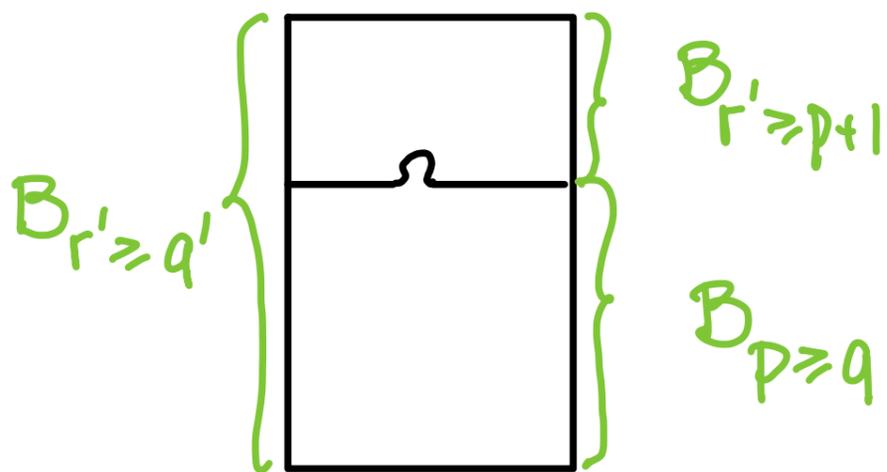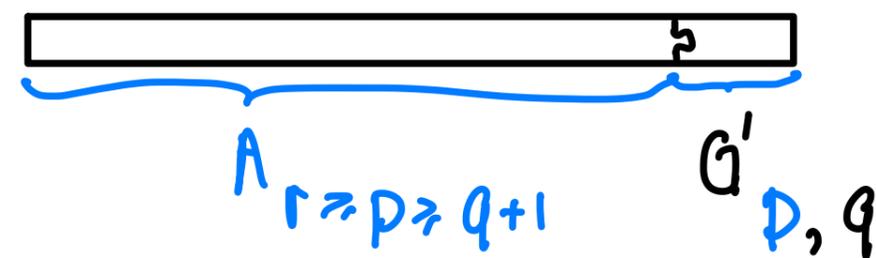
$$0 \rightarrow B_{p \geqslant q} \rightarrow B_{r' \geqslant q'} \rightarrow B_{r' \geqslant p+1} \rightarrow 0$$

These are not so hard to discover or prove now that we have the right definitions.

Then take some time to put the pieces together. Literally.

To sum up:

Depicting maps of vector spaces, abelian groups, modules etc. in a visual way makes many things much more transparent, by offloading subtle and intricate relationships onto our well-devoloped spacial intuition.

The End

Bonus Material

I. If you look closely at the E_0 page, you will see that essentially the only way to recover the filtered pieces of the cohomology of the original complex is to do precisely the actions demanded by the spectral sequence, and precisely in that (partial) order.

Second, you see precisely the information contained in each page of the spectral sequence.

To put it more precisely, there is a category $\mathcal{E}_p$ for each page p of the spectral sequence (corresponding precisely to our picture of the pth page), and functors

$$\text{Filtered Complexes} \xrightarrow{\text{Gr}} \mathcal{E}_0 \xrightarrow{d} \mathcal{E}_1 \xrightarrow{d} \mathcal{E}_2 \xrightarrow{d} \cdots$$

III. You might then notice that the categories $\mathcal{E}_p$ are all basically the same

if you carefully regrade. So we actually have a "spectral sequence page" category, and each "turning of the page" is doing the same thing.

$$\text{Filtered complexes} \xrightarrow{\ Gr\ } \text{Page category} \circlearrowleft d$$

(You need to be careful to see how to extract the "limit".)

IV. This point of view also answers a question I have always had about the precise meaning of theorems about spectral sequences.

Many theorems say "there is a spectral sequence with second page given by such-and-such, which abuts to something we want to understand."

I wanted to really understand the meaning of the theorem. We had the E_2 page, and its arrows, but there is some additional information in addition, that <u>determines</u> the arrows on the later pages. Now we know the answer. It is precisely the information in the "E_2 picture", which includes the information of which collections of pieces we are "allowed to talk about."

Here is the information in the "full" E_2 page (in our worked example).
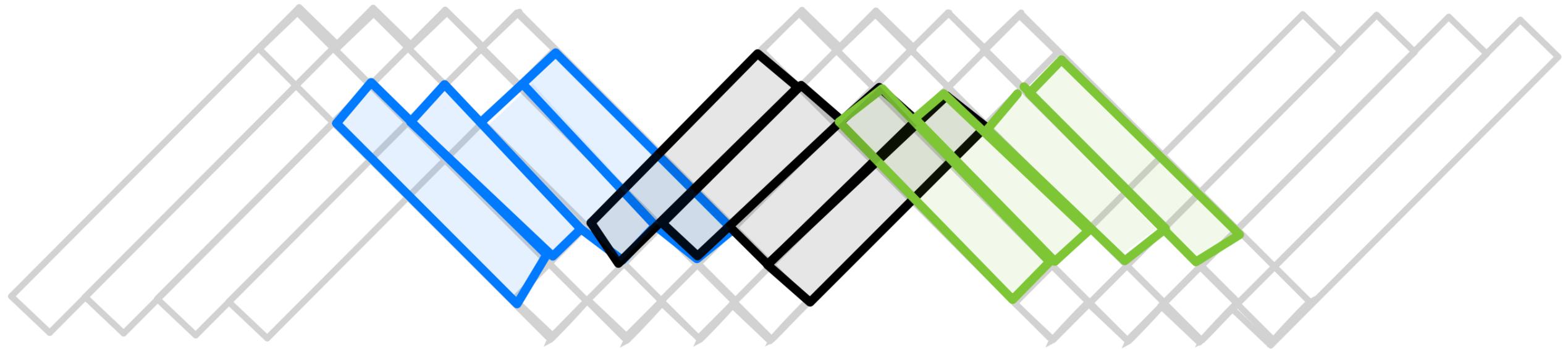
This is the information which should be in the statement of such theorems.

Ⅴ. Finally, we see easily what it is about the original filtered complex that makes the spectral sequence converge on page E_2:

A spectral sequence converges at E^2 if and only if
$$d(F^j C^n) \cap F^k C^{n+1} = d(F^{k+1} C^n) \cap F^k C^{n+1} \quad \text{for all } j \geq k+1.$$