EXPLANATION OF THE KÜNNETH FORMULA FOR QUASICOHERENT SHEAVES FROM THE POINT OF VIEW OF THE RISING SEA

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0.A. EXERCISE. Suppose X is a quasicompact separated scheme, and $j : \operatorname{Spec} A \hookrightarrow X$ is an open embedding. Show that $j_* \mathcal{O}_{\operatorname{Spec} A}$ is a quasicoherent sheaf on X, flat (over $\mathcal{O}_X$). (Partial hint: quasicoherence comes from an earlier exercise.)

Suppose further that $X = \cup_{i=1}^n U_i$ is a cover of X by a finite number of affine open sets. (We still assume X separated.) Denote the cover by $\mathfrak{U} := \{U_i\}$. As usual with Cech covers, for $I \subset \{1, \dots, n\}$, define $U_I = \cap_{i \in I} U_i$, and let $j^I : U_I \hookrightarrow X$ be the corresponding open embedding.

For the purposes of this discussion only, define the *Cech complex of sheaves* for the cover $\mathfrak{U}$ of X, denoted $C_X^\bullet(\mathfrak{U})$, by

$$0 \longrightarrow \oplus_{|I|=1} j_*^I \mathcal{O}_{U_I} \longrightarrow \oplus_{|I|=2} j_*^I \mathcal{O}_{U_I} \longrightarrow \cdots \longrightarrow \oplus_{|I|=n} j_*^I \mathcal{O}_{U_I} \longrightarrow 0.$$

Define the *augmented Cech complex of sheaves for* $X = \{U_i\}$, denoted $C_{X,aug}^\bullet(\mathcal{F})$, by prepending $\mathcal{O}_X = \oplus_{|I|=0} j_*^I \mathcal{O}_{U_I}$:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \oplus_{|I|=1} j_*^I \mathcal{O}_{U_I} \longrightarrow \oplus_{|I|=2} j_*^I \mathcal{O}_{U_I} \longrightarrow \cdots \longrightarrow \oplus_{|I|=n} j_*^I \mathcal{O}_{U_I} \longrightarrow 0.$$

(Be sure you understand the definition of the maps in these complexes.)

0.B. EXERCISE. Show that the augmented Cech complex of sheaves $C_{X,aug}^\bullet(\mathfrak{U})$ is an exact sequence of flat quasicoherent sheaves on X. (Hint: check on any affine open subset of X.)

Before proceeding further, we prove a useful homological statement. Recall that an exact sequence of flat modules remains exact upon tensoring with any other module. Recall also that an exact sequence of modules remains exact upon tensoring with any flat module (a version of the FHHF theorem). The following statement generalizes both of these naturally.

0.C. EXERCISE. Suppose $C^\bullet$ is an finite exact sequence of A-modules, and $F^\bullet$ is an finite exact sequence of *flat* A-modules. Show that the total complex of the double complex $C^\bullet \otimes_A F^\bullet$ is also an exact sequence of A-modules. (If you wish, you might show that if $C^\bullet$ is merely a complex, not necessarily exact, then $C^\bullet \otimes_A F^\bullet$ has "the same cohomology" as $C^\bullet$, further extending the FHHF theorem. You may also wish to remove the finiteness assumptions irrelevant to your solution.)

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0.D. EXERCISE. We return to the situation of Exercise 0.B. Suppose $\mathcal{F}$ is a quasicoherent sheaf (not necessarily flat) on X (which is quasicompact and separated). Show that $\mathcal{F} \otimes C^\bullet_{X,aug}(\mathfrak{U})$ is an exact sequence sequence of quasicoherent sheaves on X. (Equivalently, $\mathcal{F} \otimes C^\bullet_X(\mathfrak{U})$ is a complex, exact except at the first step, where it has kernel/cohmology sheaf canonically identified with $\mathcal{F}$.) Hint: check on each affine open subset of X.

0.1. *Remark.* Following Serre, you might interpret Exercises 0.B and 0.D in terms of partitions of unity.

We use these pleasant exact sequences to prove a form of the Künneth formula. Suppose that X and Y are both quasicompact separated k-schemes (e.g., varieties over k). Name the projection maps $\pi_X : X \times_k Y \to X$ and $\pi_Y : X \times_k Y \to Y$. Let $\mathfrak{U}_X$ and $\mathfrak{U}_Y$ be finite covers of X and Y (respectively) by affine open sets.

0.E. EXERCISE. Show that $\pi_X^* C^\bullet_{X,aug}(\mathfrak{U}_X)$ is an exact complex of flat quasicoherent sheaves on $X \times Y$.

Define $C^\bullet_X \boxtimes C^\bullet_Y := \pi_X^* C^\bullet_X(\mathfrak{U}_X) \otimes \pi_Y^* C^\bullet_Y(\mathfrak{U}_Y)$, interpreted as the total complex associated to the double complex of the right side.

0.F. EXERCISE. Show that $C^\bullet_X \boxtimes C^\bullet_Y$ is a complex of flat quasicoherent sheaves on $X \times_k Y$, exact except at the first step, where the cohomology/kernel is canonically identified with $\mathcal{O}_{X \times Y}$. (You may or may not find it helpful to prove a similarly statement for a similarly defined $C^\bullet_{X,aug} \boxtimes C^\bullet_{Y,aug}$.)

0.G. EXERCISE. Suppose now that $\mathcal{F}$ is a quasicoherent sheaf on X, and $\mathcal{G}$ is a quasicoherent sheaf on Y. Show that $(\mathcal{F} \boxtimes \mathcal{G}) \otimes (C^\bullet_X \boxtimes C^\bullet_Y)$ is a complex of quasicoherent sheaves on $X \times_k Y$, exact at the first step, where it the cohomology/kernel is canonically identified with $\mathcal{F} \boxtimes \mathcal{G}$.

0.H. EXERCISE (KÜNNETH FORMULA). By suitably identifying $(\mathcal{F} \boxtimes \mathcal{G}) \otimes (C^\bullet_X \boxtimes C^\bullet_Y)$ with $(\mathcal{F} \otimes C^\bullet_X) \boxtimes (\mathcal{G} \otimes C^\bullet_Y)$, show that for all n, $H^n(X \times_k Y, \mathcal{F} \boxtimes \mathcal{G}) = \oplus_{i+j=n} H^i(X, \mathcal{F}) \otimes_k H^j(Y, \mathcal{G})$. (Hint: Don't use spectral sequences; work directly with the complexes. Show by direct calculation that if you have two complexes of k-modules, the cohomology of the total complex of their tensor product is the "direct sum of the tensor product of their cohomologies".)

0.2. *Remarks on hypotheses.* Why did we require quasicoherence? Where did we use the fact that we were working over a field k?

0.3. *Extensions* $\star\star$. It is natural to consider the following generalization. Given quasicompact separated morphisms of schemes $\rho^X : X \to Z$ and $\rho^Y : Y \to Z$, the resulting $\rho^{X \times Y} : X \times_Z Y \to Z$, and quasicoherent sheaves $\mathcal{F}$ and X and $\mathcal{G}$ on Y, what is the relationship between $R^n \rho^{X \times Y}_* \mathcal{F} \boxtimes_Z \mathcal{G}$, $R^i \rho^X_* \mathcal{F}$ and $R^j \rho^Y_* \mathcal{G}$? By following through your proof, you may be able to extend the statement of Exercise 0.H considerably.